Technical University of Dresden
Faculty of Mathematics

University of Zagreb
Faculty of Electrical Engineering and Computing
Department of Applied Mathematics

# Workshop on

# Calculus of Variations and Applications

## Book of Abstracts

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# Speakers

Sören Bartels (University of Freiburg)

Juan Pablo Borthagaray (University of Maryland)

Kirill Cherednichenko (University of Bath)

Daniel Faraco (Autonomous University of Madrid)

Matteo Focardi (University of Florence)

Lorenzo Freddi (University of Udine)

John Gemmer (Wake Forest University)

Nikos Katzourakis (University of Reading)

Shokhrukh Kholmatov (University of Vienna)

Bernd Kirchheim (University of Leipzig)

Martin Kružík (Czech Academy of Sciences)

Raz Kupferman (Hebrew University of Jerusalem)

Roger Moser (University of Bath)

Boris Muha (University of Zagreb)

Roberto Paroni (University of Pisa)

Asaf Shachar (Hebrew University of Jerusalem)

Ian Tobasco (University of Michigan)

# Organizers

Mario Bukal (University of Zagreb)

Peter Hornung (TU Dresden)

Igor Velčić (University of Zagreb)

# Talks

## Approximating gradient flow evolutions of self-avoiding inextensible curves and elastic knots

Sören Bartels (*University of Freiburg*)

We discuss a semi-implicit numerical scheme that allows for minimizing the bending energy of curves within certain isotopy classes. To this end we consider a weighted sum of the bending energy B and the tangent-point functional TP, i.e.,

$$E(u) = \kappa \mathrm{B}(u) + \varrho \mathrm{TP}(u) = \frac{\kappa}{2} \int_I |u''(x)|^2 \,\mathrm{d}x + \varrho \iint\limits_{I\times I} \frac{\mathrm{d}x \,\mathrm{d}y}{r(u(y), u(x))^q}$$

with the *tangent-point radius* $r(u(y), u(x))$ which is the radius of the circle that is tangent to the curve $u$ at the point $u(y)$ and that intersects with $u$ in $u(x)$.

We define evolutions via the gradient flow for $E$ within a class of arclength parametrized curves, i.e., given an initial curve $u^0 \in H^2(I; \mathbb{R}^3)$ we look for a family $u : [0, T] \to H^2(I; \mathbb{R}^3)$ such that, with an appropriate inner product $(\cdot, \cdot)_X$ on $H^2(I; \mathbb{R}^3)$,

$$(\partial_t u, v)_X = -\,\delta E(u)[v], \quad u(0) = u^0,$$

subject to the linearized arclength constraints

$$[\partial_t u]' \cdot u' = 0, \quad v' \cdot u' = 0.$$

Our numerical approximation scheme for the evolution problem is specified via a semi-implicit discretization, i.e., for a step-size $\tau > 0$ and the associated backward difference quotient operator $d_t$, we compute iterates $(u^k)_{k=0,1,\dots} \subset H^2(I; \mathbb{R}^3)$ via the recursion

$$(d_t u^k, v)_X + \kappa([u^k]'', v'') = -\varrho \delta \mathrm{TP}(u^{k-1})[v]$$

with the constraints

$$[d_t u^k]' \cdot [u^{k-1}]' = 0, \quad v \cdot [u^{k-1}]' = 0.$$

The scheme leads to sparse systems of linear equations in the time steps for cubic $C^1$ splines and a nodal treatment of the constraints. The explicit treatment of the nonlocal tangent-point functional avoids working with fully populated matrices and furthermore allows for a straightforward parallelization of its computation.

Based on estimates for the second derivative of the tangent-point functional and a uniform bi-Lipschitz radius, we prove a stability result implying energy decay during the evolution as well as maintenance of arclength parametrization.

We present some numerical experiments exploring the energy landscape, targeted to the question how to obtain global minimizers of the bending energy in knot classes, so-called elastic knots. This is joint work with Philipp Reiter (University of Georgia).

## Structure-preserving FEM for the Q-tensor model of nematic liquid crystals with variable degree of orientation

Juan Pablo Borthagaray (*University of Maryland*)

In the Landau-de Gennes Q-tensor model for nematic liquid crystals with variable degree of orientation, the molecule distribution is given by a rank-one tensor and its degree of orientation. The tensor field accounts for the head-to-tail symmetry of the molecules: the states of director $n$ and $-n$ are indistinguishable. The Euler-Lagrange equations for the minimizer contain a degenerate elliptic for the rank-one tensor, which, in particular, allows for half-integer defects to have finite energy.

We present a structure-preserving discretization of the liquid crystal energy with piecewise linear finite elements that can handle the degenerate elliptic part without regularization, and show that it is consistent and stable. We prove $\Gamma$-convergence of discrete global minimizers to continuous ones as the mesh size goes to zero. We develop a quasi-gradient flow scheme for computing discrete equilibrium solutions and prove it has a strictly monotone energy decreasing property. Numerical experiments confirm the ability of the scheme to approximate configurations with defects.

## Unified approach to critical-contrast homogenisation of PDEs, and their link to time-dispersive media

Kirill Cherednichenko (*University of Bath*)

I will discuss a novel approach to critical-contrast homogenisation for periodic PDEs, based on the analysis of suitable Dirichlet-to-Neumann maps on the interfaces between different components of a composite medium. I will describe how the formulae of the theory of boundary triples, going back to the ideas of Birman, Krein and Vishik, allow one to derive explicit operator-norm resolvent asymptotics of the original family of PDEs describing the heterogeneous media. One of the key outcomes of our approach is that it relates homogenisation limits to a class of time-dispersive media, whose properties involve a memory-type convolution kernel. This is joint work with Yulia Ershova (Bath) and Alexander Kiselev (St. Petersburg).

### Magnetic Helicity and Subsolutions in MHD

Daniel Faraco (*Autonomous University of Madrid*)

In the recent years, the world of hydrodynamics have experienced a revolution due to the new method of building weak solutions created by De Lellis and Szekelyhidi for the Euler equation based on convex integration. Such solutions seems to be the best currently available model to explain the phenomena anomolous disspation of several quantities in turbulent regimes.The method yields bizzard solutions which have compact support in space and time, violating severely uniqueness of the Cauchy problem.

In this talk, I will describe how this circle of ideas interplay with Magneto Hydrodynamics, in short MHD, that is Maxwell and Euler equations coupled. At first glance, one is tempted to to believe that the results for Euler could be replicated with lenghtier and more complex computations. However, I will present a proof showing that such compactly supported solutions do not exist in 2 D MHD. In 3D, we can show that such compacty supported subsolution exists but there is an integral quantity, magnetic helicity which is preserved even for irregular solutions. We will show that it is also preserved by subsolutions and explain that has an interpretation as a compensated compactness quantity which imposes a nonlinear constraint on the space of subsolutions. This is a joint work with Sauli Lindberg.

### Existence of strong minimizers for the Griffith static fracture model in dimension two

Matteo Focardi (*University of Florence*)

We consider the Griffith fracture model in two spatial dimensions, and prove existence of strong minimizers, with closed jump set and continuously differentiable deformation fields. One key ingredient, is a generalization of the decay estimate by De Giorgi, Carriero, and Leaci to the vectorial situation. This is based on replacing the coarea formula by a method to approximate $SBD^p$ functions with small jump set by Sobolev functions and is restricted to two dimensions. The other two ingredients consist respectively in regularity results for vectorial elliptic problems of the elasticity type and in a method to approximate in energy $GSBD^p$ functions by $SBV^p$ ones. This is joint work with Sergio Conti (Universität Bonn) and F. Iurlano (Universit Paris 6).

### Elastic strings, ribbons, beams and rods: geography of a partly undiscovered world

Lorenzo Freddi (*University of Udine*)

A long and thin parallelepiped whose length is much larger than the diameter of its cross-section can be considered as the reference configuration of several different structures. It may be, for instance, an elastic string, an inextensible ribbon, a thin-walled beam, a rod or, possibly, even another kind of body.

Among such structures, each one exhibits a peculiar mechanical behavior due to the ratio $\delta$ between the length of the two sides of the rectangular cross-section and the magnitude $\varepsilon$ of the elastic energy which, in turn, depends on the applied loads.

Starting from the energy functional of 3D nonlinear elasticity and letting the sides of the cross-section go to zero, an asymptotic analysis by $\Gamma$-convergence shows that the 1D $\Gamma$-limit energy depends on the parameters $\delta$ and $\varepsilon$. Thus, it turns out that the set of limit 1D models is not a hierarchy, meant in the sense of an ordered set; in fact, they occupy different regions of a diagram $\delta - \varepsilon$ looking like a geographic map.

My talk is devoted to illustrate the geography of this world where the known regions have been discovered by various authors, namely, E. Acerbi, G. Buttazzo, D. Percivale, M.G. Mora, S. Müller, L. Scardia, R. Paroni, P. Hornung (in order of appearence) and myself, using sometimes similar and sometimes different tecniques. A small undiscovered wild region still resists to the efforts of the explorers.

### Isometric Immersions, Energy Minimization, and Self-Similar Patterns in Non-Euclidean Sheets

John Gemmer (*Wake Forest University*)

The edges of torn elastic sheets and growing leaves often display hierarchical self-similar like buckling patterns. On the one hand, such complex, self similar patterns are usually associated with a competition between two distinct energy scales, e.g. elastic sheets with boundary conditions that preclude the possibility of relieving in plane strains, or at alloy-alloy interfaces between distinct crystal structures. On the other hand, within the non-Euclidean plate theory this complex morphology can be understood as low bending energy isometric immersions of hyperbolic Riemannian metrics. In particular, many growth patterns generate residual in-plane strains which can be entirely relieved by the sheet forming part of a surface of revolution or a helix. In this talk we will show that this complex morphology (i) arises from isometric immersions (ii) is driven by a competition between the two principal curvatures, rather than between bending and stretching. We identify the key role of branch-point (or monkey-saddle) singularities, in complex wrinkling patterns within the class of finite bending energy isometric immersions. Using these defects we will give an explicit construction of strain-free embeddings of hyperbolic surfaces that are fractal like and have lower elastic energy than their smooth counterparts.

### On the existence and uniqueness of vectorial absolute minimisers in Calculus of Variations in $L^\infty$

Nikos Katzourakis (*University of Reading*)

The Calculus of Variations in $L^\infty$ has a relatively short history in Analysis. The scalar-valued theory was pioneered by the Swedish mathematician G. Aronsson in the 1960s and since then has developed enormously. The general vector-valued case delayed considerably to appear and its systematic development began in the 2010s. One of the most fundamental problems in the area which was open until today (and has been attempted by many researchers) concerned that of the title. In this talk I will discuss the first result in this direction, which is based on joint work with my research associate Giles Shaw.

### Minimizing movements for mean curvature flow of droplets with prescribed contact angle

Shokhrukh Kholmatov (*University of Vienna*)

In this talk I would like to discuss about the mean curvature motion of a droplet flowing by mean curvature on a horizontal hyperplane with a possibly nonconstant prescribed contact angle. After short introduction to GMM theory, using the solutions constructed as a limit of an approximation algorithm of AlmgrenTaylorWang and LuckhausSturzenhecker, the existence of a weak evolution will be shown. I also present various comparison results. This is a joint work with Giovanni Bellettini

### Convexity notions in the vectorial Calculus of Variations

Bernd Kirchheim (*University of Leipzig*)

There is an entire zoo of convexity concepts, which reflect lower semicontinuity or regularity properties of variational problems in higher dimensions, but all boil down to ordinary convexity when domain or target are one-dimensional. We are going to present less and more recent results, stating that such notions do agree if we restrict the class of functionals considered or their domains of definition.

## A phase-field approach to Eulerian interfacial energies

Martin Kružík (*Czech Academy of Sciences*)

We analyze a phase-field approximation of a sharp-interface model for two phase materials which consists of a polyconvex bulk strain energy density and interface-polyconvex energy assigned to interfaces separating phases. The distinguishing feature of the model rests in the fact that the interfacial term is Eulerian in nature, and it is defined on the deformed configuration. We discuss a functional frame allowing for existence of phase-field minimizers and $\Gamma$-convergence to the sharp-interface limit. As a by-product, we provide additional details on the admissible sharp-interface configurations with respect to the analysis. This is a joint work with D. Grandi (Ferrara), E. Mainini (Genova), and U. Stefanelli (Vienna).

## The bending energy of bucked edge-dislocations

Raz Kupferman (*Hebrew University of Jerusalem*)

The study of elastic membranes carrying topological defects has a longstanding history, going back at least to the 1950s. When allowed to buckle in three-dimensional space, membranes with defects can totally relieve their in-plane strain, remaining with a bending energy, whose rigidity modulus is small compared to the stretching modulus. It was suggested in the 1980s that a disc endowed with a single edge dislocation can totally relieve its stretching energy on the expense of a bending energy, whose magnitude is independent of the size of the system. In this lecture, I will show that this is not true: the minimum bending energy associated with strain-free configurations diverges logarithmically with the size of the system.

## A variational problem in $L^\infty$ involving the Laplacian

Roger Moser (*University of Bath*)

Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function. Consider $u\colon \Omega \to \mathbb{R}$ and suppose that we want to minimise a functional of the form $E(u) = \operatorname{ess\,sup} |F(\,\cdot\,, \Delta u)|$ under given boundary data, prescribing $u$ and its first derivative on $\partial\Omega$. Under quite natural assumptions on $F$ and $\Omega$, it turns out that we have a unique minimiser, which is quite unexpected for a variational problem of this sort. Moreover, we have a lot of information about its structure. This is joint work with Nikos Katzourakis (Reading).

## Rigorous derivation of a sixth order thin film equation

Boris Muha (*University of Zagreb*)

In this talk we will study a linear 3D/3D fluid-structure interaction (FSI) between a thin layer of a viscous fluid and a thin elastic body. First, suitable a priori estimates in terms of thickness of the fluid layer and the elastic body, which are both small parameters, will be derived. Using the obtained estimates we will identify the scaling properties of the physical parameters which give rise to a sixth-order thin film equation, which describes the evolution of the thin elastic body interacting with the thin layer of the fluid. We will analyze the convergence of solutions as small parameter (thickness of the domain) tends to zero. In the second part of this talk we will discuss extensions to the non-linear case. More precisely, starting from the moving boundary FSI problem we will derive the nonlinear sixth order thin film equation.

This is joined work with M. Bukal, University of Zagreb.

## A REBO-Potential-Based Model for Graphene Bending by Γ-Convergence

Roberto Paroni (*University of Pisa*)

We present an atomistic to continuum model for a graphene sheet undergoing bending, within the small displacements approximation framework. Under the assumption that the atomic interactions are governed by a harmonic approximation of the 2nd-generation Brenner REBO (reactive empirical bond-order) potential, involving the first, second and third nearest neighbors of any given atom, we determine the variational limit of the energy functionals. It turns out that the Gamma-limit depends on the linearized mean and Gaussian curvatures. If some specific contributions in the atomic interaction are neglected, the variational limit is non-local.

The talk is based on joint work with C. Davini and A. Favata.

## On the Role of Curvature in Non-Euclidean Thin Elastic Bodies

Asaf Shachar (*Hebrew University of Jerusalem*)

A major topic in elasticity is the study of thin elastic bodies, e.g. plates, shells and rods. In incompatible elasticity, when the metric of the thin elastic body is non-Euclidean (and smooth), the elastic energy of a plate or a shell of thickness $h$ scales like $h^2$ or $h^4$, depending on the metric, and for rods it always scales like $h^4$. In this talk I will present a general result for thin bodies of arbitrary dimension and co-dimension, that relates the elastic energy scaling to the Riemannian curvature of the body along its mid-surface. This includes the scalings of plates, shells and rods as special cases.

This is a joint work with Cy Maor.

### The Cost of Crushing: Curvature-Driven Wrinkling of Thin Elastic Shells

Ian Tobasco (*University of Michigan*)

How much energy does it cost to stamp a curved elastic shell flat? Motivated by recent experiments on the wrinkling patterns formed by thin shells floating on a water bath, we develop a rigorous method via Gamma-convergence for evaluating the cost of crushing to leading order in the shell's thickness and other small parameters. The experimentally observed patterns involve regions of well-defined wrinkling alongside totally "disordered" regions in which no single direction of wrinkling is preferred. Our goal is to explain the appearance and lack thereof of such wrinkling domains". The basic mathematical objects of study are short maps from the mid-shell into the plane, describing the overall shape of the shell. To such short maps we assign an energy proportional to the area covered in the plane, so that a sort of surface tension effect determines the cost of stamping and, oftentimes, the wrinkling domains. Somewhat surprisingly, we find that only part of this cost comes from the bare surface tension of the exposed liquid-vapor interface, while a second part comes from the cost of superimposing low amplitude, high frequency perturbations on top of the limiting maps. Resolving this limiting model in many cases, we offer new explicit predictions and comparisons with experiment.

# List of participants

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Sören Bartels (University of Freiburg)
Juan Pablo Borthagaray (University of Maryland)
Mario Bukal (University of Zagreb)
Marin Bužančić (University of Zagreb)
Kirill Cherednichenko (University of Bath)
Tvrtko Dorešić (University of Paris-Sud)
Peter Eberhard (Technical University of Dresden)
Daniel Faraco (Autonomous University of Madrid)
Matteo Focardi (University of Florence)
Lorenzo Freddi (University of Udine)
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## Notes

Zagreb