Homogenization result for two-phase two-component flow in porous medium

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Abstract

We derive a homogenization result for the system that describes a two-phase two-component flow in heterogeneous porous media, written in fully equivalent global pressure formulation. A global pressure, an artificial variable that allows us to decouple original equations, is used as introduced in [3]. Under some realistic assumptions on the data, we obtain a nonlinear homogenized problem by using the techniques of two-scale convergence.

Contents

1 Introduction

Problems of multiphase multicomponent flow in porous media appear in many applications, as in petroleum engineering, nuclear waste storage, CO_2 sequestration and many others. For example, in nuclear waste management one often meets flow of two phases, liquid and gas, that are composed of two components, water and pure hydrogen H_2. In that application it is of great importance to know the pressure of the gas phase in a host rock, to avoid overpressurization.

Multiphase multicomponent flow in porous media is usually modeled by a system of nonlinear partial differential equations that represent mass conservation of each component, combined with initial and boundary conditions. These equations are strongly coupled, and in obtaining the solution of such a system numerical simulations play a special role. For modelling such flow problems, there are always multiple length scales in the physical coefficients for the governing equations. On the other hand, the size of the repository model prohibits a full fine scale simulation over many time steps, even with the advent of modern computers and parallel computing technology. Therefore, a compromise has to be made between desired accuracy and available computer resources. The standard compromise is to upscale the coefficients which allows the use of a coarse computational grid. The upscaling or homogenization of multiphase flow through heterogeneous porous media has been a problem of interest for many years and many methods have been developed.

Previous results on the homogenization for two-phase two-component flow have been derived in [2]. In this paper they used a concept of global pressure as introduced in [19] originally for the incompressible case.

The goal of this paper is to rigorously derive the homogenized or effective model for two-phase two-component flow in porous medium in a global pressure formulation that was introduced in [3] and further studied in [7], [8], [4], [5], and which is fully equivalent to the original phase equations formulation. Compared to results of [17], [18], [21], this new formulation allows us to avoid additional regularizations, which makes our proof less technical. We consider a system with diffusivity terms in both component equations. In our work we pose realistic assumptions on data, similar to those in [21]. In particular, we cover more general case of Henry law and ideal gas law since we do not assume strict positivity of the gass density. In this general case, the degeneracy caused by possible vanishing of the gas pressure combined with the degeneracy due to the saturation complicates the analysis of the system.

The rest of the paper is organized as follows. In Section 2 we introduce the notation, we formulate the microscopic problem and we state the assumptions on data. For completeness, we quote the existence result of weak solutions of the microscopic problem obtained in [7]. The main result of this paper is the homogenized model presented in Section 3. The a priori estimates with respect to the space and time variables of weak solutions of the microscopic problem are obtained in Section 4. Next, Section 5 is devoted to the compactness results needed to obtain the effective equations. Finally, our homogenization result is proven in Section 6, using the two-scale convergence technique.

2 Mathematical formulation

We start from a microscopic model which is defined on a domain with periodic microstructure. Let us define such a domain more precisely. We consider a reservoir $\Omega \subset \mathbb{R}^d$, $d = 2,3$, to be a bounded, connected, Lipschitz domain with a periodic structure. The reservoir boundary is divided in two parts, $\partial \Omega = \Gamma_{inj} \cup \Gamma_{imp}$, where Γ_{inj} denotes the injection boundary, and Γ_{imp} denotes the impervious one. In such a geometrical configuration of the reservoir, we take that the porosity and the absolute permeability tensor are rapidly oscillating functions of the microscopic scale $y = x/\varepsilon$, where x is the macroscopic scale and $\varepsilon > 0$ is a small parameter which characterizes the periodicity of the blocks. Namely, let $\Phi^\varepsilon(x) = \Phi(x/\varepsilon)$ and $\mathbb{K}^\varepsilon(x) = \mathbb{K}(x/\varepsilon)$ be the porosity and the absolute permeability of the porous medium, where Φ and $\mathbb{K}$ are Y-periodic functions of y. Further, the unit cell is noted by $Y = (0,1)^d$, the time interval of interest is $(0,T)$ and $Q_T = \Omega \times (0,T)$.

A porous medium is assumed to be filled with a fluid composed of two phases, liquid (denoted by index l) and gas denoted by g. Moreover, the fluid is considered as a mixture of two components: a liquid component which does not evaporate, denoted by upper index w (suggesting water), and a low-soluble component which is present mostly in the gas phase, denoted by upper index h (suggesting hydrogen). The porous medium is assumed to be rigid and in a thermal equilibrium, while the liquid component is assumed incompressible.

Let $S_l^\varepsilon = S_l^\varepsilon(x,t)$, $S_g^\varepsilon = S_g^\varepsilon(x,t)$ be the saturations of the wetting and the non-wetting phases, respectively; $p_l^\varepsilon = p_l^\varepsilon(x,t)$, $p_g^\varepsilon = p_g^\varepsilon(x,t)$ are the pressures of the wetting and the non-wetting phases, respectively; $\lambda_l = \lambda_l(S_l^\varepsilon)$, $\lambda_g = \lambda_g(S_g^\varepsilon)$ be the relative mobilities of the wetting and the non-wetting phases, respectively; and ρ_l^ε, ρ_g^ε be the mass densities of the wetting and the non-wetting phases, respectively. For each phase $\sigma \in \{l,g\}$, the phase volumetric fluxes $\mathbf{q}_\sigma^\varepsilon$ are defined by the Darcy-Muskat law:

$$\begin{aligned}
\mathbf{q}_l{}^\varepsilon &= -\lambda_l \mathbb{K}^\varepsilon(\mathbf{x})\left(\nabla p_l^\varepsilon - \rho_l \mathbf{g}\right), \\
\mathbf{q}_g{}^\varepsilon &= -\lambda_g \mathbb{K}^\varepsilon(\mathbf{x})\left(\nabla p_g^\varepsilon - \rho_g \mathbf{g}\right),
\end{aligned} \tag{1}$$

where $\mathbf{g}$ is the gravity acceleration. The porous medium is saturated by the two phases: the phase saturations satisfy

$$S_l^\varepsilon + S_g^\varepsilon = 1. \tag{2}$$

The phase pressures are related through a given capillary pressure law:

$$p_c(S_g^\varepsilon) = p_g^\varepsilon - p_l^\varepsilon. \tag{3}$$

We note that here p_c is a strictly decreasing function of liquid saturation, $p_c'(S_l) < 0$. We will take that $p_c(S_l = 1) = P_0$. The phase mobilities will be assumed to depend on capillary pressure, not on saturation. Therefore it holds $\lambda_g(P_0) = 0$ which corresponds to the case $\lambda_g(S_g = 0) = 0$ if the gas mobility is considered as a function of saturation. In this work we will not consider the cases of $S_l = 0$ or $S_g = 0$.

As we have already mentioned, in the gas phase we neglect the liquid component vaporization and therefore the gas mass density depends only on the gas pressure:

$$\rho_g^\varepsilon = \hat{\rho}_g(p_g^\varepsilon). \tag{4}$$

In the case of the ideal gas law, (4) becomes $\hat{\rho}_g(p_g) = C_v p_g$ with a constant C_v.

The quantity of the gas component dissolved in the liquid is described by the mass concentration $\rho_l^{h,\varepsilon}$ which gives the mass of dissolved gas component in the volume of the liquid mixture. For simpler notation we will denote $\rho_l^{h,\varepsilon}$ by u^ε. From the assumption of thermodynamic equilibrium we conclude that

$$u^\varepsilon = \hat{u}(p_g^\varepsilon). \tag{5}$$

The function $\hat{u}$ can be a linear function if the Henry law is applicable. We suppose that the function $p_g^\varepsilon \mapsto \hat{u}(p_g^\varepsilon)$ is defined and invertible on $[0,\infty)$ and therefore we can express the gas pressure as a function of $\hat{u}$,

$$p_g^\varepsilon = \hat{p}_g(u^\varepsilon), \tag{6}$$

where $\hat{p}_g$ is the inverse of $\hat{u}$.

For liquid density, due to hypothesis of small solubility and liquid incompressibility we assume a constant liquid component mass concentration, i.e.:

$$\rho_l^{w,\varepsilon} = \rho_l^{std}, \tag{7}$$

where ρ_l^{std} is the standard liquid component mass density (a constant). The liquid mass density is then: $\rho_l^\varepsilon = \rho_l^{std} + u^\varepsilon$.

The mass conservation law for each component gives then the following system of differential equations:

$$\rho_l^{std}\Phi^\varepsilon \frac{\partial S_l^\varepsilon}{\partial t} + \mathrm{div}\left(\rho_l^{std}\mathbf{q}_l{}^\varepsilon + \mathbf{j}_l^{w,\varepsilon}\right) = 0, \tag{8}$$

$$\Phi^\varepsilon \frac{\partial}{\partial t}\left(u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon\right) + \mathrm{div}\left(u^\varepsilon \mathbf{q}_l{}^\varepsilon + \rho_g^\varepsilon \mathbf{q}_g{}^\varepsilon + \mathbf{j}_l^{h,\varepsilon}\right) = 0, \tag{9}$$

where $\mathbf{j}_l^{w,\varepsilon}, \mathbf{j}_l^{h,\varepsilon}$ are the diffusive flux terms in the liquid phase (see equations (10)).

The diffusive fluxes in the liquid phase are given by the Fick law through the gradient of the mass fractions $u^\varepsilon/\rho_l^\varepsilon$ and $\rho_l^{w,\varepsilon}/\rho_l^\varepsilon$ as in [14] and see also [13]:

$$\mathbf{j}_l^{h,\varepsilon} = -\Phi^\varepsilon S_l^\varepsilon D\rho_l^\varepsilon \nabla(u^\varepsilon/\rho_l^\varepsilon), \quad \mathbf{j}_l^{w,\varepsilon} = -\Phi^\varepsilon S_l^\varepsilon D\rho_l^\varepsilon \nabla(\rho_l^{w,\varepsilon}/\rho_l^\varepsilon), \tag{10}$$

where D is a molecular diffusion coefficient of dissolved gas in the liquid phase, possibly corrected by the tortuosity factor of the porous medium (see [13]). Note that it holds $\mathbf{j}_l^{w,\varepsilon} + \mathbf{j}_l^{h,\varepsilon} = 0$.

In the model described here both phases will always be present, which means that we will consider capillary pressure to be a positive function defined on $\langle 0, 1\rangle$. In such two-phase regions one can select as primary variables S_l (or p_c) and p_g, for example. Our approach will be, similarly as in [15], [21] to select as primary variables p_g and a global pressure introduced in [3], [6], [8]. One can calculate u from relation (6) and for calculating the capillary pressure one needs to obtain the value of the liquid pressure $p_l(p, p_g)$. After that we will obtain the liquid saturation by inverting the capillary pressure, $S_l = p_c^{-1}(p_g - p_l)$.

Let us now rewrite the equations (8) and (9), including expressions of Darcy velocities and Fick's law.

$$\Phi^\varepsilon \frac{\partial S_l^\varepsilon}{\partial t} - \mathrm{div}\left(\lambda_l(S_l^\varepsilon)\mathbb{K}^\varepsilon(\nabla p_l^\varepsilon - \rho_l^\varepsilon \mathbf{g}) - \Phi^\varepsilon S_l^\varepsilon \frac{1}{\rho_l^\varepsilon}D\nabla u^\varepsilon \right) = 0, \tag{11}$$

$$\Phi^\varepsilon \frac{\partial}{\partial t}\left(u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon \right) - \mathrm{div}\left(u^\varepsilon \lambda_l(S_l^\varepsilon)\mathbb{K}^\varepsilon \left(\nabla p_l^\varepsilon - \rho_l^\varepsilon \mathbf{g} \right) + \rho_g^\varepsilon \lambda_g(S_g^\varepsilon)\mathbb{K}^\varepsilon \left(\nabla p_g^\varepsilon - \rho_g^\varepsilon \mathbf{g} \right) \right)$$
$$- \mathrm{div}\left(\Phi^\varepsilon S_l^\varepsilon \frac{\rho_l^{std}}{\rho_l^\varepsilon}D\nabla u^\varepsilon \right) = 0. \tag{12}$$

Let us first consider in equations (11), (12) the liquid pressure p_l^ε and the gas pressure p_g^ε as independent variables. The mass concentration u^ε and the gas mass density ρ_g^ε are then calculated from the gas pressure as $u^\varepsilon = \hat{u}(p_g^\varepsilon)$ and $\rho_g^\varepsilon = \hat{\rho}_g(p_g^\varepsilon)$. The saturation S_l^ε is obtained from the phase pressures by the capillary pressure law.

We denote the term under the time derivative in the equation (12) by $r_g^\varepsilon := u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon$.

In the following section we will introduce new primary variable, the global pressure, which will be used instead of the liquid pressure.

2.1 Global pressure formulation

The motivation for the introduction of the global pressure is to decouple the original phase system equations. It is achieved by writing the total flux $\mathbb{Q}_t^\varepsilon = \mathbb{Q}^{w,\varepsilon} + \mathbb{Q}^{h,\varepsilon} := \rho_l^\varepsilon \mathbf{q}_l^\varepsilon + \rho_g^\varepsilon \mathbf{q}_g^\varepsilon$ in a form of the Darcy-Muskat law. In this work we use a global pressure formulation that was introduced in [3] and further studied in [7], [8], [4], [5], and which is fully equivalent to the original phase equations formulation. The microscopic equations for our two-phase two-component flow in porous media in this fully equivalent global pressure formulation are as follows:

$$\Phi^\varepsilon \frac{\partial}{\partial t}(\rho_l^{std}S_l^\varepsilon) - \mathrm{div}(\frac{\rho_l^{std}\omega^\varepsilon \lambda}{\rho_l^\varepsilon}\mathbb{K}^\varepsilon \nabla p^\varepsilon - \frac{\rho_l^{std}\rho_g^\varepsilon \lambda_g}{\rho_l^\varepsilon}\mathbb{K}^\varepsilon \nabla p_g^\varepsilon)$$
$$+ \mathrm{div}(b^{w,\varepsilon}\mathbb{K}^\varepsilon \mathbf{g}) + \mathrm{div}\left(\Phi^\varepsilon \frac{S_l^\varepsilon \rho_l^{std}}{\rho_l^\varepsilon}D\nabla u^\varepsilon \right) = 0, \tag{13}$$

$$\Phi^\varepsilon \frac{\partial}{\partial t}(u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon) - \mathrm{div}\left(\frac{u^\varepsilon}{\rho_l^\varepsilon}\omega^\varepsilon \lambda \mathbb{K}^\varepsilon \nabla p^\varepsilon + \frac{\rho_l^{std}\rho_g^\varepsilon \lambda_g}{\rho_l^\varepsilon}\mathbb{K}^\varepsilon \nabla p_g^\varepsilon - b^{h,\varepsilon}\mathbb{K}^\varepsilon \mathbf{g} \right)$$
$$- \mathrm{div}\left(\Phi^\varepsilon \frac{S_l^\varepsilon \rho_l^{std}}{\rho_l^\varepsilon}D\nabla u^\varepsilon \right) = 0. \tag{14}$$

Here a new pressure variable p^ε, called the global pressure, is introduced, and we express the liquid pressure as function of p^ε: $p_l^\varepsilon = \tilde{p}_l(p^\varepsilon, P_c^\varepsilon)$. This function satisfies (see [6])

$$\nabla \tilde{p}_l(p^\varepsilon, P_c^\varepsilon) + f_g \nabla P_c^\varepsilon = \tilde{\omega}(p^\varepsilon, P_c^\varepsilon)\nabla p^\varepsilon, \tag{15}$$

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where

$$\frac{\partial \tilde{p}_g^\varepsilon}{\partial p^\varepsilon}(p^\varepsilon, P_c^\varepsilon) = \frac{\partial \tilde{p}_l^\varepsilon}{\partial p^\varepsilon}(p^\varepsilon, P_c^\varepsilon) = \tilde{\omega}(p^\varepsilon, P_c^\varepsilon). \tag{16}$$

The function $\tilde{\omega}$ can be explicitly calculated, it is well defined, bounded and strictly positive for $P_c^\varepsilon \geq P_0$ and $p^\varepsilon \in \mathbb{R}$, and therefore the application $p^\varepsilon \mapsto \tilde{p}_l(p^\varepsilon, P_c^\varepsilon)$ is invertible for all $P_c^\varepsilon \geq P_0$. There exist constants $\omega_m, \omega_M > 0$ such that $\omega_m \leq \tilde{\omega}(p^\varepsilon, P_c^\varepsilon) \leq \omega_M$.

In the obtained equation we consider the coefficients as functions of P_c^ε instead of the saturation directly, simply by relations $f(P_c^\varepsilon) = f(S_l^\varepsilon(P_c^\varepsilon))$. The value of the saturation in the time derivative term can be calculated from the relation $S_l^\varepsilon = p_c^{-1}(P_c^\varepsilon)$.

In the system (13)-(14) the total mobility $\lambda(p_g^\varepsilon, P_c^\varepsilon)$, the fractional flow functions of the gas and liquid phases $f_g(p_g^\varepsilon, P_c^\varepsilon)$, $f_l(p_g^\varepsilon, P_c^\varepsilon)$, and the mean density function $\rho(p_g^\varepsilon, P_c^\varepsilon)$ are given as follows:

$$\lambda = \rho_l^\varepsilon \lambda_l(P_c^\varepsilon) + \rho_g^\varepsilon \lambda_g(P_c^\varepsilon) \tag{17}$$

$$f_g = \frac{\rho_g^\varepsilon \lambda_g(P_c^\varepsilon)}{\lambda(p_g^\varepsilon, P_c^\varepsilon)}, \quad f_l = \frac{\rho_l^\varepsilon \lambda_l(P_c^\varepsilon)}{\lambda(p_g^\varepsilon, P_c^\varepsilon)} \tag{18}$$

$$\rho = \frac{(\rho_l^\varepsilon)^2 \lambda_l(P_c^\varepsilon) + (\rho_g^\varepsilon)^2 \lambda_g(P_c^\varepsilon)}{\lambda(p_g^\varepsilon, P_c^\varepsilon)} \tag{19}$$

$$b^{w,\varepsilon} = \rho_l^{std} \rho_l^\varepsilon \lambda_l(P_c^\varepsilon) \tag{20}$$

$$b^{h,\varepsilon} = u\rho_l^\varepsilon \lambda_l(P_c^\varepsilon) + (\rho_g^\varepsilon)^2 \lambda_g(P_c^\varepsilon). \tag{21}$$

Note that $f_g + f_l = 1$. Therefore we can express capillary pressure as a function of the global pressure and the gas pressure, and this leads us to a system with two primary unknowns p_g^ε and p^ε. All the coefficients in equations depend on p^ε and p_g^ε, through $P_c(p^\varepsilon, p_g^\varepsilon)$, so we set $\omega^\varepsilon = \tilde{\omega}(p, P_c(p, p_g))$.

To the system (13)-(14) we add initial and boundary conditions.

Boundary conditions: We set

$$p_g^\varepsilon = 0, \quad p^\varepsilon = 0 \quad \text{on } \Gamma_{inj} \times (0, T) \tag{22}$$

$$\mathbb{Q}^{w,\varepsilon} \cdot \mathbf{n} = 0, \quad \mathbb{Q}^{h,\varepsilon} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{imp} \times (0, T) \tag{23}$$

where $\mathbf{n}$ is the outward pointing unit normal on $\partial\Omega$.

Initial conditions are given by

$$p_g^\varepsilon(x, 0) = p_g^0(x), \quad p^\varepsilon(x, 0) = p^0(x) \quad \text{in } \Omega. \tag{24}$$

Remark 1 *In apriori estimates we will use also the auxiliary variable θ^ε (as in [7]) which is introduced by*

$$\theta^\varepsilon = \beta(P_c^\varepsilon) = \int_0^{P_c^\varepsilon} \sqrt{\lambda_l(s)\lambda_g(s)} \, ds, \tag{25}$$

and which is well defined since β is strictly increasing.

Remark 2 *Note that in the definition of the global pressure we have imposed the equality*

$$\rho_l^\varepsilon \lambda_l \nabla p_l^\varepsilon + \rho_g^\varepsilon \lambda_g \nabla p_g^\varepsilon = \omega^\varepsilon \lambda \nabla p^\varepsilon. \tag{26}$$

Next we present the assumptions on the data which will ensure the existence of a weak solution of the system (13)-(14).

(A.1) The porosity Φ belongs to $L^\infty(\Omega)$, and there exist constants, $\phi_M \geq \phi_m > 0$, such that $\phi_m \leq \Phi(x) \leq \phi_M$ a.e. in Ω. The diffusion coefficient D belongs to $L^\infty(\Omega)$, and there exists a constant $D_0 > 0$ such that $D(x) \geq D_0$ a.e. in Ω.

(A.2) The permeability tensor $\mathbb{K}$ belongs to $(L^\infty(\Omega))^{d \times d}$, and there exist constants $k_M \geq k_m > 0$, such that for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$ it holds:

$$k_m |\xi|^2 \leq \mathbb{K}(x)\xi \cdot \xi \leq k_M |\xi|^2.$$

(A.3) The function $\hat{u}(p_g)$ is increasing C^1 function from $[0, +\infty)$ to $[0, +\infty)$ and $\hat{u}(0) = 0$. There exist constants $u_{max} > 0$, $u_0 > 0$ and $M_g > 0$ such that for all $\sigma \geq 0$ it holds,

$$|\hat{u}(\sigma)| \leq u_{max}, \quad u_0 \leq \hat{u}'(\sigma) \leq M_g.$$

For $\sigma \leq 0$ we extend $\hat{u}(\sigma)$ as a smooth, suficiently small, bounded function having global C^1 regularity. The main low solubility assumption is that the constant M_g is sufficiently small.

(A.4) Function $\hat{\rho}_g(p_g)$ is a C^1 non decreasing function on $[0, \infty)$, and there exist constants $\rho_M > 0$ and $m_g > 0$ such that for all $p_g \geq 0$ it holds

$$0 \leq \hat{\rho}_g(p_g) \leq \rho_M, \quad m_g p_g \leq \hat{\rho}_g(p_g), \quad \hat{\rho}_g(0) = 0, \quad \int_0^1 \frac{d\sigma}{\hat{\rho}_g(\sigma)} < \infty.$$

For $\sigma \leq 0$ we set $\hat{\rho}_g(\sigma) = 0$ for all $\sigma \leq 0$.

(A.5) Relative mobilities λ_l, λ_g are defined as $\lambda_l(S_l) = kr_l(S_l)/\mu_l$ and $\lambda_g(S_l) = kr_g(S_l)/\mu_g$ where the constants $\mu_l > 0$ and $\mu_g > 0$ are the liquid and the gas viscosities, and $kr_l(S_l)$, $kr_g(S_l)$ are the relative permeability functions, satisfying $\lambda_l, \lambda_g \in C([0, 1])$, $\lambda_l(0) = 0$ and $\lambda_g(1) = 0$; the function λ_l is a non decreasing and λ_g is non increasing function of S_l. Moreover, there exist a constant $\lambda_m > 0$ such that for all $S_l \in [0, 1]$

$$\lambda_m \leq \lambda := \rho_l \lambda_l + \rho_g \lambda_g$$

We assume also that there exists a constant $a_l > 0$ such that for all $S_l \in [0, 1]$:

$$a_l S_l^2 \leq \lambda_l(S_l). \tag{27}$$

(A.6) The capillary pressure function, $p_c \in C^1(0,1)$, is monotone decreasing function satisfying $p_c(S_l) > 0$ for $S_l \in (0,1)$ and $p'_c(S_l) \leq -M_0 < 0$ for $S_l \in (0,1)$ and some constant $M_0 < 0$. There exists a positive constant M_{p_c}

$$\int_0^1 p_c(s)\,ds = M_{p_c} < +\infty, \quad \lim_{S_l \to 0+} S_l p_c(S_l) = 0. \tag{28}$$

The inverse functions p_c^{-1} is extended as $p_c^{-1}(\sigma) = 1$ for $\sigma \leq 0$.

(A.7) There exist $C > 0$ and $\tau \in (0,1)$ such that for all $S_1, S_2 \in [0,1]$

$$C \left| \int_{S_1}^{S_2} \sqrt{\lambda_l(s)\lambda_g(s))}\,ds \right|^\tau \geq |S_1 - S_2|. \tag{29}$$

Let us recall that the primary variables are p and p_g. The secondary variables are the functions u, ρ_g, S_l and S_g and p_l, defined as $u = \hat{u}(p_g)$, $\rho_g = \hat{\rho}_g(p_g)$, $P_c = P_c(p, p_g)$, $S_l = p_c^{-1}(P_c)$ and $S_g = 1 - S_l$, $p_l = p_g - P_c$. By (A.3) and (A.4) the functions u and ρ_g are bounded and for S_l, due to (A.6), we have

$$0 < S_l \leq 1. \tag{30}$$

Variational formulation is obtained by standard arguments. We define

$$V = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D\}$$

The following existence result has been proven in [26]:

Theorem 1 *Let (A.1)-(A.7) hold and assume $(p^0, p_g^0) \in L^2(\Omega) \times L^2(\Omega)$, $p_g^0 \geq 0$. Let $\varepsilon > 0$, then there exist functions p^ε and p_g^ε satisfying*

$$p_l^\varepsilon \in L^2(Q_T), \quad p^\varepsilon, p_g^\varepsilon \in L^2(0,T;V),$$
$$\Phi^\varepsilon \partial_t(u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon), \ \Phi^\varepsilon \partial_t S_l^\varepsilon \in L^2(0,T;V'),$$

such that: for all $\varphi \in L^2(0,T;V)$

$$\int_0^T \langle \Phi^\varepsilon \frac{\partial S_l^\varepsilon}{\partial t}, \varphi \rangle dt + \int_{Q_T} [\frac{\omega^\varepsilon \lambda}{\rho_l^\varepsilon} \mathbb{K}^\varepsilon \nabla p^\varepsilon - \frac{\rho_g^\varepsilon \lambda_g}{\rho_l^\varepsilon} \mathbb{K}^\varepsilon \nabla p_g^\varepsilon - \Phi^\varepsilon S_l^\varepsilon \frac{1}{\rho_l^\varepsilon} \hat{u}'(p_g^\varepsilon) D \nabla p_g^\varepsilon] \cdot \nabla \varphi dx dt$$
$$= \int_{Q_T} \rho_l^\varepsilon \lambda_l \mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla \varphi dx dt; \tag{31}$$

for all $\psi \in L^2(0,T;V)$

$$\int_0^T \langle \Phi^\varepsilon \frac{\partial}{\partial t} (u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon), \psi \rangle dt$$
$$+ \int_{Q_T} \left(\frac{u^\varepsilon}{\rho_l^\varepsilon} \omega^\varepsilon \lambda \mathbb{K}^\varepsilon \nabla p^\varepsilon + \frac{\rho_l^{std} \rho_g^\varepsilon \lambda_g}{\rho_l^\varepsilon} \mathbb{K}^\varepsilon \nabla p_g^\varepsilon + \Phi^\varepsilon S_l^\varepsilon \frac{\rho_l^{std}}{\rho_l^\varepsilon} \hat{u}'(p_g^\varepsilon) D \nabla p_g^\varepsilon \right) \cdot \nabla \psi dx dt \tag{32}$$
$$= \int_{Q_T} b^{h,\varepsilon} \mathbb{K}^\varepsilon \mathbf{g} \cdot \nabla \psi dx dt.$$

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Furthermore, for all $\psi \in V$ the functions

$$t \mapsto \int_\Omega \Phi^\varepsilon S_l^\varepsilon \psi dx, \quad t \mapsto \int_\Omega \Phi((u^\varepsilon - \rho_g^\varepsilon)S_l^\varepsilon + \rho_g^\varepsilon)\psi dx$$

are continuous in $[0,T]$ and the initial condition is satisfied in the following sense:

$$\left(\int_\Omega \Phi^\varepsilon S_l^\varepsilon \psi dx\right)(0) = \int_\Omega \Phi^\varepsilon s_0 \psi dx,$$

$$\left(\int_\Omega \Phi^\varepsilon (u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon)\psi dx\right)(0) = \int_\Omega \Phi(\hat{u}(p_g^0)s_0 + \hat{\rho}_g(p_g^0)(1-s_0))\psi dx,$$

for all $\psi \in V$, where $s_0 = p_c^{-1}(P_c^\varepsilon(p^0, p_g^0))$.

3 Homogenization result

As we have already pointed out, we study the asymptotic behaviour of the solution to the problem (13)-(14), (22)-(23), (24) as $\varepsilon \to 0$. In particular, we are going to show that the effective model valid in Q_T reads as follows.

$$\langle\Phi\rangle\frac{\partial}{\partial t}(\rho_l^{std}S_l) - \mathrm{div}(\frac{\rho_l^{std}\omega\lambda}{\rho_l}\mathbb{K}^*\nabla p - \frac{\rho_l^{std}\rho_g\lambda_g}{\rho_l}\mathbb{K}^*\nabla p_g)$$
$$+ \mathrm{div}(b^w\mathbb{K}^*\mathbf{g}) + \mathrm{div}\left(\langle\Phi\rangle\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right) = 0, \tag{33}$$

$$\langle\Phi\rangle\frac{\partial}{\partial t}(uS_l + \rho_g S_g) - \mathrm{div}\left(\frac{u}{\rho_l}\omega\lambda\mathbb{K}^*\nabla p + \frac{\rho_l^{std}\rho_g\lambda_g}{\rho_l}\mathbb{K}^*\nabla p_g - b^h\mathbb{K}^*\mathbf{g}\right)$$
$$- \mathrm{div}\left(\langle\Phi\rangle\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right) = 0. \tag{34}$$

Here $\langle u\rangle$ denotes the mean value of the function u over the unit cell Y and the homogenized tensor $\mathbb{K}^*$ is given by

$$\mathbb{K}^*\mathbf{e}_i = \int_Y \mathbb{K}(\mathbf{e}_i + \nabla_y\chi_i(y))dy. \tag{35}$$

Here, $\chi_i(y)$ (for $i = 1, \ldots, d$) is a solution of the cell problem

$$\begin{cases} - \mathrm{div}_y\left(\mathbb{K}(y)(\mathbf{e}_i + \nabla_y\chi_i(y))\right) = 0 \text{ in } Y, \\ \chi_i(y) \, Y - \text{ periodic}, \end{cases} \tag{36}$$

with $\mathbf{e}_i$ being the unit vector in the i-th direction.

The coefficients b^w, b^h are given by

$$b^w = \rho_l^{std} \rho_l \lambda_l(P_c) \tag{37}$$

$$b^h = u\rho_l \lambda_l(P_c) + (\rho_g)^2 \lambda_g(P_c). \tag{38}$$

The boundary conditions for the system (33)-(34) are

$$p_g = 0, \quad p = 0 \quad \text{on } \Gamma_{inj} \times (0,T) \tag{39}$$

$$\mathbb{Q}^w \cdot \mathbf{n} = 0, \quad \mathbb{Q}^h \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{imp} \times (0,T) \tag{40}$$

where

$$\mathbb{Q}^w = -\rho_l \lambda_l \mathbb{K}^*(\mathbf{x})\left(\nabla p_l - \rho_l \mathbf{g}\right),$$

$$\mathbb{Q}^h = -\rho_g \lambda_g \mathbb{K}^*(\mathbf{x})\left(\nabla p_g - \rho_g \mathbf{g}\right).$$

The initial conditions for the system (33)-(34) read

$$p_g(x,0) = p_g^0(x), \quad p(x,0) = p^0(x) \quad \text{in } \Omega. \tag{41}$$

Now we can state the main result of this paper.

Theorem 2 *Let (A.1)-(A.7) hold. Let $(p^\varepsilon, p_g^\varepsilon)$ be a weak solution of the problem* (13)-(14), (22)-(23), (24). *Then, up to a subsequence, it holds*

$$p^\varepsilon \rightharpoonup p \text{ weakly in } L^2(0,T;H^1(\Omega)),$$

$$p_g^\varepsilon \rightharpoonup p_g \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q_T,$$

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)),$$

$$S_l^\varepsilon \to S_l \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,$$

$$r_g^\varepsilon = u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon \to r_g := uS_l + \rho_g S_g \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,$$

$$\nabla p^\varepsilon(x,t) \xrightarrow{2s} \nabla p(x,t) + \nabla_y w_1(x,y,t),$$

$$\nabla p_g^\varepsilon(x,t) \xrightarrow{2s} \nabla p_g(x,t) + \nabla_y w_2(x,y,t).$$

The pair (p, p_g) is a weak solution of the problem (33)-(41).

Here $\xrightarrow{2s}$ denotes the two-scale convergence whose definition is recalled at the beginning of Section 6. Theorem 2 is proven in Section 6.

4 A priori estimates

In this section we develop the a priori estimates that will serve to prove Theorem 2.

A priori estimates that will be used in the proof of the main result are based on the following equality, which can be easily checked:

$$\rho_g^\varepsilon \lambda_g \mathbb{K}^\varepsilon \nabla p_g^\varepsilon \cdot \nabla p_g^\varepsilon + \rho_l^\varepsilon \lambda_l \mathbb{K}^\varepsilon \nabla p_l^\varepsilon \cdot \nabla p_l^\varepsilon = \lambda \omega^2 \mathbb{K} \nabla P \cdot \nabla P + \frac{\rho_g \rho_l}{\lambda} \mathbb{K} \nabla \theta \cdot \nabla \theta.$$

Motivated by [21], we introduce the test functions

$$\varphi^\varepsilon = p_l^\varepsilon - N(p_g^\varepsilon), \quad \psi = M(p_g^\varepsilon),$$

where

$$M(p_g^\varepsilon) = \int_0^{(p_g^\varepsilon)^+} \frac{1}{\hat{\rho}_g(\sigma)} d\sigma, \quad N(p_g^\varepsilon) = \int_0^{(p_g^\varepsilon)^+} \frac{\hat{u}(\sigma)}{\hat{\rho}_g(\sigma)} d\sigma. \tag{42}$$

The functions M and N are extended by zero for negative pressures.
We set

$$\mathscr{E}(p^\varepsilon, p_g^\varepsilon) = S_l^\varepsilon \left(\hat{u}(p_g^\varepsilon) M(p_g^\varepsilon) - N(p_g^\varepsilon) \right) + S_g^\varepsilon \left(\hat{\rho}_g(p_g^\varepsilon) M(p_g^\varepsilon) - p_g^\varepsilon \right) - \int_0^{S_l^\varepsilon} p_c(s) ds,$$

where the dependence of $\mathscr{E}$ on global pressure is given through $S_l^\varepsilon(p^\varepsilon, p_g^\varepsilon)$.

Remark 3 *From Lemma 6 in [21] we have for $p_g \geq 0$:*

$$-M_{p_c} \leq \mathscr{E}(p, p_g) \leq C(|p_g| + 1). \tag{43}$$

Note that the test functions $p_l^\varepsilon - N(p_g^\varepsilon)$ and $M(p_g^\varepsilon)$ satisfy the following identity:

$$\frac{\partial S_l^\varepsilon}{\partial t} \left(p_l^\varepsilon - N(p_g^\varepsilon) \right) + \frac{\partial}{\partial t} \left(u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon \right) M(p_g^\varepsilon) = \frac{\partial}{\partial t} \mathscr{E}(p^\varepsilon, p_g^\varepsilon).$$

By using the functions M, N and $\mathscr{E}$ the following result can be proved as in [21].

Lemma 1 *Let the assumptions (A.1)-(A.7) be fulfilled and let the initial conditions p^0 and p_g^0 be such that $\mathscr{E}(p^0, p_g^0) \in L^1(\Omega)$. Then there is a constant C independent of ε such that each solution of (31), (32) satisfies:*

$$\int_{Q_T} \left\{ \lambda_l(S_l^\varepsilon) |\nabla p_l^\varepsilon|^2 + \lambda_g(S_l^\varepsilon) |\nabla p_g^\varepsilon|^2 + |\nabla u^\varepsilon|^2 \right\} \leq C, \tag{44}$$

$$\int_{Q_T} \left\{ |\nabla p^\varepsilon|^2 + |\nabla p_g^\varepsilon|^2 + |\nabla u^\varepsilon|^2 \right\} \leq C, \tag{45}$$

$$\left\| \partial_t (\Phi^\varepsilon [u^\varepsilon S_l^\varepsilon + \rho_g^\varepsilon S_g^\varepsilon]) \right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\| \partial_t (\Phi^\varepsilon S_l^\varepsilon) \right\|_{L^2(0,T;H^{-1}(\Omega))} \leq C. \tag{46}$$

5 A compactness result

In this section we establish the compactness results for the families $\{S^\varepsilon\}_{\varepsilon>0}$ and $\{r_g^\varepsilon\}_{\varepsilon>0}$ in the space $L^2(Q_T)$. The proof is based on the compactness lemma from [2], which we now recall.

Lemma 2 *(Lemma 4.2, [2]) Let the function $\Phi = \Phi(y)$ be a Y-periodic function, $\Phi \in L^\infty(Y)$, and there are positive constants ϕ_m, ϕ_M such that $0 < \phi_m \leq \Phi(y) \leq \phi_M < 1$ a.e. in Y, $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ and let $\{v^\varepsilon\}_{\varepsilon>0} \subset L^2(Q_T)$ be a family of functions satisfying the properties:*

1. *the function v^ε is uniformly bounded in the space $L^\infty(Q_T)$, i.e. $0 \leq v^\varepsilon \leq C$;*

2. *there exists a function ϖ such that $\varpi(\xi) \to 0$ as $\xi \to 0$ and the following inequality holds true:*
$$\int_{Q_T} |v^\varepsilon(x+h,t) - v^\varepsilon(x,t)|^2 dxdt \leq C\varpi(|h|);$$

3. *the function v^ε is such that*
$$\|\frac{\partial}{\partial t}(\Phi^\varepsilon v^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$

Then the family $\{v^\varepsilon\}_{\varepsilon>0}$ is a compact set in $L^2(Q_T)$.

By using Lemma 2 and uniform in ε a priori estimates from Lemma 1, we can prove the following results.

Proposition 1 *Assuming (A.1)-(A.7), the families $\{S^\varepsilon\}_{\varepsilon>0}$ and $\{r_g^\varepsilon\}_{\varepsilon>0}$ are compact in $L^2(Q_T)$.*

Corollary 1 *There exist functions $S \in L^2(Q_T)$ and $r_g \in L^2(Q_T)$ such that, up to a subsequence,*

$$S^\varepsilon \to S \text{ a. e. in } Q_T, \tag{47}$$
$$r_g^\varepsilon \to r_g \text{ a. e. in } Q_T. \tag{48}$$

6 The proof of the homogenization result

The goal of this section is to rigorously justify the convergence results for the homogenized problem (13)-(14), (22)-(23), (24) given by Theorem 2. In order to pass to the limit as $\varepsilon \to 0$ in the weak formulation (31)-(32), we use the a priori estimates and the compactness results of the previous two sections. Moreover, we use the two-scale convergence technique, see e.g. [1].

We begin this section with some definitions and standard results on two-scale convergence taken from [1], slightly modified for the case of homogenization with a parameter t (like for example in [9]). However, we point out that these modifications do not affect the proofs from [1] in any essential way.

By $\mathscr{D}(Q)$ we denote the space of infinitely smooth and compactly supported functions in Q with values in $\mathbb{R}$, by $C_p^\infty(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^d$ which are periodic of period Y and by $H_p^1(Y)$ the space of functions in $H_{loc}^1(\mathbb{R}^d)$ which are periodic of period Y.

Definition 1 *An admissible test function is any function $\varphi \in L^2(Q_T; C_p^\infty(Y))$ which satisfies*

$$\lim_{\varepsilon \to 0} \int_{Q_T} |\varphi(x, \frac{x}{\varepsilon}, t)|^2 dxdt = \int_{Q_T \times Y} |\varphi(x, y, t)|^2 dxdydt.$$

Definition 2 *A sequence of functions $v^\varepsilon \in L^2(Q_T)$ two-scale converges to $v \in L^2(Q_T \times Y)$, denoted by $v^\varepsilon(x, t) \overset{2s}{\rightharpoonup} v(x, y, t)$, if for any admissible test function $\varphi(x, y, t)$,*

$$\lim_{\varepsilon \to 0} \int_{Q_T} v^\varepsilon(x, t)\varphi(x, \frac{x}{\varepsilon}, t)dxdt = \int_{Q_T \times Y} v(x, y, t)\varphi(x, y, t)dydxdt.$$

We denote by $H = \{u \in H_p^1(Y) : \int_Y u dy = 0\} = H_p^1(Y)/\mathbb{R}$.

Theorem 3 *Let (u_ε) be a bounded sequence in $L^2(0, T; H^1(\Omega))$ with a subsequence that converges weakly to a limit u in $L^2(0, T; H^1(\Omega))$. Then, along this subsequence, (u_ε) two-scale converges to $u(x, t)$. Also, there exists a function u_1 in $L^2(Q_T; H)$ such that, up to a subsequence, (∇u_ε) two-scale converges to $\nabla_x u + \nabla_y u_1$.*

Theorem 4 *([11, Theorem 9]) Let $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let (u_ε) be a sequence in $L^p(Q_T)$ which two-scale converges to u. Then*

$$\lim_{\varepsilon \to 0} \int_{Q_T} u_\varepsilon(x, t)\psi(x, \frac{x}{\varepsilon}, t)dxdt = \int_{Q_T} \int_Y u(x, y, t)\psi(x, y, t)dydxdt,$$

for every ψ of the form $\psi(x, y, t) = \psi_1(x, t)\psi_2(y)$, $\psi_1 \in L^{rq}(Q_T)$, $\psi_2 \in L_p^{sq}(Y)$ with $1 \leq r, s \leq +\infty$ and such that $\frac{1}{r} + \frac{1}{s} = 1$.

The uniform estimates and the compactness results for the functions p^ε, p_g^ε, S^ε and r_g^ε obtained in the previous sections will be used for passing to the limit in the weak formulation (31)-(32). More precisely, the a priori estimates from Lemma 1 imply that there exist $p, p_g \in L^2(0, T; V)$ such that, up to a subsequence,

$$p^\varepsilon \rightharpoonup p \text{ in } L^2(0, T; V), \quad p_g^\varepsilon \rightharpoonup p_g \text{ in } L^2(0, T; V).$$

From the same a priori estimates and Theorem 3, there exist functions

$$w_1(x, y, t), w_2(x, y, t) \in L^2(Q_T; H)$$

such that

$$\nabla p^\varepsilon(x, t) \overset{2s}{\rightharpoonup} \nabla p(x, t) + \nabla_y w_1(x, y, t), \tag{49}$$

$$\nabla p_g^\varepsilon(x, t) \overset{2s}{\rightharpoonup} \nabla p_g(x, t) + \nabla_y w_2(x, y, t). \tag{50}$$

The rest of the proof consists of passing to the limit as $\varepsilon \to 0$ in the weak formulation (13)-(14). It is achieved in a standard way by considering for the both equations (13), (14) the test function of the form

$$\varphi(x, \frac{x}{\varepsilon}, t) = \varphi(x, t) + \varepsilon \zeta(x, \frac{x}{\varepsilon}, t) = \varphi(x, t) + \varepsilon \zeta_1(x, t)\zeta_2(\frac{x}{\varepsilon}),$$

where $\varphi \in \mathscr{D}(Q_T)$, $\zeta_1 \in \mathscr{D}(Q_T)$, $\zeta_2 \in C_p^\infty(Y)$ and employing the a priori estimates, the strong convergence results, the two-scale convergence results and Lebesgue's theorem, with the latter result being applicable due to the almost everywhere convergence from Corollary 1, the continuity and the boundedness of the coefficients.

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