STRONG TRACES TO DEGENERATE PARABOLIC EQUATIONS

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Abstract. We prove existence of strong traces at $t = 0$ for quasi-solutions to the degenerate parabolic equations under non-degeneracy assumptions. In order to solve the problem, we introduce a defect measure type functional and combine it with the blow up method.

1. Introduction

In the current contribution, we consider the advection diffusion equation:

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathfrak{f}(u) = D^2 \cdot A(u), \tag{1}$$

where $\mathfrak{f} \in C^1(\mathbf{R}; \mathbf{R}^d)$, $A \in C^2(\mathbf{R}; M^{d\times d})$ and $D^2 \cdot A(u) = \sum\limits_{k,j} \partial^2_{x_k x_j} A_{kj}(u)$ and A is symmetric. Usually, it is written in the form (non convenient for us at the moment)

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathfrak{f}(u) = \operatorname{div}_{\mathbf{x}}(a(u)\nabla u),$$

where $a(\lambda) = A'(\lambda)$ and $\operatorname{div}_{\mathbf{x}}(a(u)\nabla u) = \sum\limits_{k,j}^{d} \partial_{x_j}(a_{kj}(u)\partial_{x_k} u)$. Given equation is very important and it describes phenomena containing the combined effects of nonlinear convection, degenerate diffusion, and nonlinear reaction. More precisely, the equation describes a flow governed by

- the convection effects (bulk motion of particles) which are represented by the first order terms;
- diffusion effects which are represented by the second order term and the matrix $A'(\lambda) = [a_{ij}(\lambda)]_{i,j=1,\dots,d}$ describes directions and intensities of the diffusion and it satisfies for a symmetric matrix $\sigma = [\sigma_{kj}]_{k,j=1,\dots,d} \in C^1(\mathbf{R}; M^{d\times d})$:

$$\langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi}\rangle = \sum_{k,j}^{d} a_{kj}(\lambda)\xi_j\xi_k = \sum_{k=1}^{d}\left(\sum_{j=1}^{d} \sigma_{jk}(\lambda)\xi_j\right)^2.$$

The equation is degenerate in the sense that the matrix $a(\lambda) = A'(\lambda)$ can be equal to zero in some direction. Roughly speaking, if this is the case (i.e. if for some vector $\boldsymbol{\xi} \in \mathbf{R}^d$ we have $\langle A'(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi}\rangle = 0$), then diffusion effects do not exist for the state λ in the direction $\boldsymbol{\xi}$.

The equation appears in a broad spectrum of applications, such as e.g. flow in porous media [17], sedimentation-consolidation processes [9] and many others which we omit here (see the Introduction from [12] for more details). Existence and uniqueness for the Cauchy problem corresponding to (1) is well established in quite general situations [10, 11, 12]. The question of existence of strong traces for

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entropy solutions to (1) is however intact. Let us recall that the function $u = u(t, \mathbf{x})$ has the trace $u_0 = u_0(\mathbf{x})$ at $t = 0$ if $L^1_{loc} - \lim_{t\to 0} u(t, \cdot) = u_0(\cdot)$. More precisely, we shall use the following definition.

Definition 1. Let $u \in L^p(\mathbf{R}^+ \times \mathbf{R}^d)$. A locally integrable function u_0 defined on $\mathbf{R}^d$ is called the strong trace of u at $t = 0$ if for any relatively compact set $K \subset\subset \mathbf{R}^d$ it holds

$$\lim_{t\to 0} \|u(t, \cdot) - u_0\|_{L^1(K)} = 0. \tag{2}$$

The strong traces appeared in the context of limit of hyperbolic relaxation toward scalar conservation laws [27, 35]. Also, they appeared to be very useful specially related to the uniqueness of solution to scalar conservation laws with discontinuous flux (see very restrictive list [2, 3, 13] and references therein).

One of the first results concerning the existence of traces was proved in [36] for entropy solutions to scalar conservation laws [20] where the basic technique for the proof – the blow up technique – was introduced. The results are further extended in [30] for quasi-solutions to scalar conservation laws by combining the blow up techniques and the H-measures. All the mentioned results were confined on homogeneous scalar conservation laws. We extended them in [1] on heterogeneous ultra-parabolic equations under special assumptions. Let us remark in passing that existence of traces for entropy solutions for general multi-dimensional scalar conservation laws is still open.

As for the (entropy) solutions to degenerate parabolic equations (see e.g. [10, 11, 12, 37]), there are no results for traces either in homogeneous or heterogeneous setting. An obvious problem is inadequacy of the standard blow-up technique which involves scaling of the variables. Namely, if we are in the hyperbolic setting we use the scaling $(t, \mathbf{x}) \mapsto (\varepsilon t, \varepsilon \mathbf{x})$ (the same with respect to both variables) [36] while in the (ultra) parabolic setting, we need $(t, \mathbf{x}) \mapsto (\varepsilon t, \sqrt{\varepsilon}\bar{\mathbf{x}} + \varepsilon\hat{\mathbf{x}})$, $\mathbf{x} = (\bar{\mathbf{x}}, \hat{\mathbf{x}})$ [1]. This clearly causes problems if the equations changes type.

As we shall see, the defect measures techniques that we are going to introduce here will be able to overcome mentioned obstacle. They are inspired by works [23] where the defect measures are introduced, and [6, 34] where the one-scale H-measures were introduced. Actually, the one scale H-measures are generalization of the H-measures (or micro-local defect measures) [18, 33], while the H-measures are introduced as a generalization of the defect measures from [23]. More precisely, while defect measures take into account only the space variable, the H-measures take into account space and dual variables and they are thus "finer" than the defect measures. The standard example is the following sequence

$$u_n(x) = \exp(-ikx), \quad x \in (-\pi, \pi), \quad k \text{ is fixed},$$

and the corresponding defect measure is defined as the weak limit along a subsequence of $(u_n^2)_{n\in\mathbf{N}}$ in the space of Radon measures $\mathcal{M}([-\pi, \pi])$. Interestingly, for any $k \in \mathbf{N}$, the defect measure is the same and it is equal to the Lebesgue measure which means that the oscillation effects were not taken into account. This is not the case with the H-measures which "see" the oscillation effects as well. In the recent couple of years, several variants of the H-measures and their generalization appeared (see e.g. [4, 5, 21, 26]), and one of the last are one-scale H-measures which, roughly speaking, allow test functions to depend on the parameter $n \in \mathbf{N}$. By combining this idea with the defect measures, we will get the tools necessary to prove existence of the strong traces.

The paper is organized as follows.

In Section 2, we recall and adapt variants of the defect measures that we are going to use. Also, we introduce the non-degeneracy conditions. Remark that such conditions are standard in approaches that involve kinetic formulations [21, 24, 32] i.e. reduction of nonlinear equations to transport type equations [14, 15, 22, 31] and then using the velocity averaging results [21, 24, 31, 32]. In the final section, we prove existence of strong traces for quasi-solutions to (1) under the non-degeneracy conditions.

2. Auxiliary statements, notions and notations

In this section, we shall introduce the notion of the defect measures – the basic tools that we are going to use. The defect measures introduced in [23] describe loss of the strong L^2_{loc} compactness due to concentration effects, but they are insensitive to oscillation effects. However, if we take the Fourier transform $(\mathcal{F}(u_n))$ of the sequence (u_n) generating the defect measure, and consider the defect measure generated by $(\mathcal{F}(u_n))$, we actually get an object describing the oscillation effects. In this contribution, we know that we cannot have the concentration effects (we are dealing with bounded sequences), so we can consider behaviour of the sequences of interest only in the dual space. This in particular avoids question of extension of bilinear functionals which typically appears in the frame of H-measures (see [22, 25] in a more general situation).

In order to introduce the necessary tools, we need the notion of the Fourier multiplier operator.

Definition 2. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the Fourier and inverse Fourier transform. The mapping $\mathcal{A}_\psi : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$, $\psi \in L^\infty(\mathbf{R}^d)$, defined by

$$\mathcal{A}_\psi(u) = \mathcal{F}^{-1}(\psi(\boldsymbol{\xi})\mathcal{F}(u))$$

is called the Fourier multiplier operator with the symbol ψ.

The fact that the symbol ψ is bounded provides L^2-continuity of the mapping $\mathcal{A}_\psi$. Question of necessary and sufficient conditions for L^p-continuity, $p \neq 2$, of the mapping $\mathcal{A}_\psi$ is still open. However, there exist a few criterions giving necessary conditions for the L^p-continuity. One of them is the Marcinkiewicz multiplier theorem one of whose consequences we shall need later.

Theorem 3. [19, Theorem 5.2.4.]. *The multiplier operator $\mathcal{A}_\psi$ with the symbol ψ is continuous as the mapping $L^r(\mathbf{R}^d) \to L^r(\mathbf{R}^d)$, $r > 1$, if it holds*

$$\Big|\boldsymbol{\xi}^\alpha D^\alpha \psi(\boldsymbol{\xi})| \leq C, \quad \boldsymbol{\xi} \in \mathbf{R}^d \backslash \textit{coordinate axis}$$

for every multi-index $\alpha \in \mathbf{N}_0^d$ such that $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d \leq d$.

Here, we used the notations $\boldsymbol{\xi}^\alpha = \prod_{i=1}^d \xi_i^{\alpha_i}$ and $D^\alpha = \prod_{i=1}^d \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i}$.

Now, we can introduce the defect measure that we are going to use. In the sequel

$$t \in [0,\infty) = \mathbf{R}^+, \ \mathbf{x} \in \mathbf{R}^d \text{ i.e. } (t,\mathbf{x}) \in \mathbf{R}^d_+, \quad \text{and} \quad (\mathbf{y},\lambda) \in \mathbf{R}^{d+1}.$$

The following theorem holds.

Theorem 4. *Let $q : \mathbf{R} \to M^{d\times d}$ be a non-negative matrix valued continuous function and let (ζ_n) be a positive sequence of real numbers. Let (u_n) be a sequence bounded in $L^2(\mathbf{R}^d_+ \times \mathbf{R}^{d+1})$, $p \geq 2$, uniformly compactly supported with respect to $\lambda \in \mathbf{R}$ in $K_\lambda \subset\subset \mathbf{R}$. Let (v_n) be a sequence bounded in $L^2(\mathbf{R}^d_+ \times \mathbf{R}^d_{\mathbf{y}})$. Then, there exists a measure $\mu \in \mathcal{M}(\mathbf{R}^{d+1} \times [-1,1]^{d+1})$ such that for every $\psi \in C_0(\mathbf{R}^{d+1} \times [-1,1]^{d+1})$, it holds along a subsequence*

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbf{R}^d_+ \times \mathbf{R}^{d+1}} u_n(t,\mathbf{x},\mathbf{y},\lambda)\,\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})\,dtd\mathbf{x}d\mathbf{y}d\lambda \qquad (3)\\
&= \langle \mu, \psi(\mathbf{y},\lambda,\mathbf{w})\rangle, \quad (\mathbf{y},\lambda) \in \mathbf{R}^{d+1},\ \mathbf{w} \in [-1,1]^{d+1},
\end{aligned}
$$

where, for every fixed $(\mathbf{y},\lambda) \in \mathbf{R}^{d+1}$, the multiplier operator $\mathcal{A}_{\psi(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})} : L^2(\mathbf{R}^d_+) \to L^2(\mathbf{R}^{1+d})$ is continuous and $\overline{\mathcal{A}}_\psi$ is the complex conjugate of $\mathcal{A}$.

Remark 5. Remark that if we add a test function φ depending on $(t,\mathbf{x})$ in (3) we get a variant of the H-measure. If $\theta_n \equiv 0$ and (u_n) is independent of λ, then we have a variant introduced in [18, 33]. If $q(\lambda) = Id$ and $\theta_n = \frac{1}{n}$ then we get one-scale H-measures [6, 34]. However, the situation is not that simple since we would get a bilinear functional on $C_0(\mathbf{R}^{1+d}) \times C_0(\mathbf{R}^{d+1} \times [-1,1]^{d+1})$ in (3), and a nontrivial question of extension to $C_0(\mathbf{R}^{1+d} \times \mathbf{R}^{d+1} \times [-1,1]^{d+1})$ would arise.

Proof: Consider the sequence of mappings $\mu_n : C_0(\mathbf{R}^{d+1} \times [-1,1]^{d+1}) \to \mathbf{C}$ defined by

$$
\psi \mapsto \int_{\mathbf{R}^d_+ \times \mathbf{R}^{d+1}} u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1)}\right)}(v_n)(t,\mathbf{x},\mathbf{y})\,dtd\mathbf{x}d\mathbf{y}d\lambda \qquad (4)
$$

which is uniformly bounded in $\mathcal{M}(\mathbf{R}^{d+1} \times [-1,1]^{d+1})$.

Indeed, using the Plancherel theorem, we have for any $\psi \in C_0(\mathbf{R}^{d+1} \times [-1,1]^{d+1})$

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^d_+ \times \mathbf{R}^{d+1}} u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})\,dtd\mathbf{x}d\mathbf{y}d\lambda\Big| \qquad (5)\\
&= \Big|\int_{\mathbf{R}^{d+1}_{(\mathbf{y},\lambda)} \times \mathbf{R}^{d+1}_{(\xi_0,\xi)}} \psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right) \times\\
&\qquad\qquad \times \mathcal{F}(u_n)(\xi_0,\boldsymbol{\xi},\mathbf{y},\lambda)\overline{\mathcal{F}(v_n)}(\xi_0,\boldsymbol{\xi},\mathbf{y})\,d\xi_0 d\boldsymbol{\xi} d\mathbf{y} d\lambda\Big|\\
&\leq \int_{\mathbf{R}^{d+1}_{(\mathbf{y},\lambda)}} \sup_{\mathbf{w}\in[-1,1]^{d+1}} |\psi(\mathbf{y},\lambda,\mathbf{w})| \times\\
&\qquad\qquad \times \|\mathcal{F}(u_n)(\cdot,\cdot,\mathbf{y},\lambda)\|_{L^2(\mathbf{R}^{d+1})}\,\|\mathcal{F}(v_n)(\cdot,\cdot,\mathbf{y})\|_{L^2(\mathbf{R}^{d+1})}\,d\mathbf{y}d\lambda.
\end{aligned}
$$

From here, using uniform compact support of (u_n) with respect to λ and the Plancherel theorem, we immediately get

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^d_+ \times \mathbf{R}^{d+1}} u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})\,dtd\mathbf{x}d\mathbf{y}d\lambda\Big|\\
&\leq \|\psi\|_{C_0(\mathbf{R}^{d+1}\times[-1,1]^{d+1})}\mathrm{meas}(K_\lambda)^{1/2}\|u_n\|_{L^2(\mathbf{R}^d_+\times\mathbf{R}^{1+d})}\|v_n\|_{L^2(\mathbf{R}^d_+\times\mathbf{R}^d_{\mathbf{y}})}.
\end{aligned}
$$

Thus, we see that the sequence (μ_n) is bounded in $\mathcal{M}(\mathbf{R}^{d+1}\times[-1,1]^{d+1})$ (with the bound equal to $\sup\limits_{n\in\mathbf{N}}\operatorname{meas}(K_\lambda)^{1/2}\|u_n\|_{L^2(\mathbf{R}^d_+\times\mathbf{R}^{1+d})}\|v_n\|_{L^2(\mathbf{R}^d_+\times\mathbf{R}^d_{\mathbf{y}})}$). Using the weak compactness for the space of Radon measures, we conclude that there exists a subsequence (μ_n) (not relabelled) such that (3) holds. ❑

In the next theorem we shall refine the object from the previous theorem i.e. we shall prove that in a special situation μ can be more precisely estimated. To this end, we denote

$$\Lambda_0(\boldsymbol{\xi})=\{\lambda\in\mathbf{R}:\ \langle a(\lambda)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\rangle=0\}$$

$$\Lambda^\varepsilon=\{\lambda\in\mathbf{R}:\ \langle a(\lambda)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\rangle>\varepsilon\}.$$

We have the following theorem.

Theorem 6. *Assume that, in addition to the assumptions of Theorem 4, there exists a sequence δ_n such that*

(i) $\lim\limits_{n\to\infty}\frac{\sigma_n}{\delta_n^2}=\infty$;

(ii) *the sequences $(\tilde u_n)=(u_n(\delta_n t,\delta_n\mathbf{x},\mathbf{y},\lambda))$ and $(\tilde v_n)=(v_n(\delta_n t,\delta_n\mathbf{x},\mathbf{y}))$ are bounded in $L^1\cap L^2(\mathbf{R}^d_+\times\mathbf{R}^{d+1})$ and $L^1\cap L^2(\mathbf{R}^d_+\times\mathbf{R}^d)$, respectively.*

Then for the H-,=measure μ defined in Theorem 4, any bounded function $\psi:\mathbf{R}^{d+1}\times\mathbf{R}^{d+1}\to\mathbf{R}$ and any relatively compact $K_\lambda\subset\subset\mathbf{R}$, it holds

$$\Big|\langle\psi,\mu\rangle\Big|\leq C\Big(\sup_{\xi_0,\boldsymbol{\xi},\mathbf{y}}\|\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+1}\right)\|_{L^2(K_\lambda\cap\Lambda_0(\boldsymbol{\xi}))}+\sup_{\xi_0,\boldsymbol{\xi},\mathbf{y}}\|\psi(\mathbf{y},\lambda,0)\|_{L^2(K_\lambda\cap\Lambda^0(\boldsymbol{\xi}))}\Big) \tag{6}$$

i.e. we have for a measure $\tilde\mu\in\mathcal{M}(\mathbf{R}^{d+1}\times\mathbf{R}^{d+1})$

$$\langle\psi,\mu\rangle=\langle\tilde\mu,\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+1}\right)\chi_{\Lambda_0(\boldsymbol{\xi})}+\psi(\mathbf{y},\lambda,0)\chi_{\Lambda^0(\boldsymbol{\xi})}\rangle \tag{7}$$

where χ_A is the characteristic function of the set A.

Proof: We have

$$\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}}u_n(t,\mathbf{x},\mathbf{y},\lambda)\,\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})dtd\mathbf{x}d\mathbf{y}d\lambda \tag{8}$$

$$\leq\Big|\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^d}\int_{\Lambda_0\cap K_\lambda}u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})dtd\mathbf{x}d\mathbf{y}d\lambda\Big|$$

$$+\Big|\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^d}\int_{\Lambda^0\cap K_\lambda}u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\zeta_n\langle q(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(v_n)(t,\mathbf{x},\mathbf{y})dtd\mathbf{x}d\mathbf{y}d\lambda\Big|$$

The first integral on the right-hand side above clearly provides the first summand on the right-hand side of (6). As for the second integral, let us introduce here the change of variables

$$(\xi_0,\boldsymbol{\xi})=\frac{1}{\delta_n}(\tilde\xi_0,\tilde{\boldsymbol{\xi}}).$$

We get

$$
\begin{aligned}
&\Big|\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^d}\int_{K_\lambda\cap\Lambda^0(\boldsymbol\xi} u_n(t,\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol\xi)}{|(\xi_0,\boldsymbol\xi)|+\zeta_n\langle q(\lambda)\boldsymbol\xi,\boldsymbol\xi\rangle}\right)}(v_n)(t,\mathbf{x},\mathbf{y})dtd\mathbf{x}d\mathbf{y}d\lambda\Big| \qquad (9)\\
&=\Big|\int\limits_{\mathbf{R}^d_{\mathbf{y}}\times\mathbf{R}^{d+1}_{(\tilde\xi_0,\tilde\xi)}}\frac{1}{\delta_n^{(d+1)/2}}\overline{\mathcal{F}(v_n)}(\frac{\tilde\xi_0}{\delta_n},\frac{\tilde{\boldsymbol\xi}}{\delta_n},\mathbf{y})\times\\
&\quad\times\int_{K_\lambda\cap\Lambda^0(\tilde{\boldsymbol\xi}}\psi\left(\mathbf{y},\lambda,\frac{(\tilde\xi_0,\tilde{\boldsymbol\xi})}{|(\tilde\xi_0,\tilde{\boldsymbol\xi})|+\frac{\zeta_n}{\delta_n^2}\langle q(\lambda)\tilde{\boldsymbol\xi},\tilde{\boldsymbol\xi}\rangle}\right)\frac{1}{\delta_n^{(d+1)/2}}\mathcal{F}(u_n)(\frac{\tilde\xi_0}{\delta_n},\frac{\tilde{\boldsymbol\xi}}{\delta_n},\mathbf{y},\lambda)d\lambda d\tilde\xi_0 d\tilde{\boldsymbol\xi}d\mathbf{y}\Big|\\
&\leq\int\limits_{\mathbf{R}^d_{\mathbf{y}}\times\mathbf{R}^{d+1}_{(\tilde\xi_0,\tilde\xi)}}\frac{1}{\delta_n^{(d+1)/2}}\big|\overline{\mathcal{F}(v_n)}(\frac{\tilde\xi_0}{\delta_n},\frac{\tilde{\boldsymbol\xi}}{\delta_n},\mathbf{y})\big|\,\|\psi\left(\mathbf{y},\cdot,\frac{(\tilde\xi_0,\tilde{\boldsymbol\xi})}{|(\tilde\xi_0,\tilde{\boldsymbol\xi})|+\frac{\zeta_n}{\delta_n^2}\langle q(\cdot)\tilde{\boldsymbol\xi},\tilde{\boldsymbol\xi}\rangle}\right)\|_{L^2(K_\lambda\cap\Lambda^0(\tilde{\mathbf{x}}))}\times\\
&\qquad\times\|\frac{1}{\delta_n^{(d+1)/2}}\mathcal{F}(u_n)(\frac{\tilde\xi_0}{\delta_n},\frac{\tilde{\boldsymbol\xi}}{\delta_n},\mathbf{y},\cdot)\|_{L^2(K_\lambda\cap\Lambda^0(\tilde{\boldsymbol\xi}))}d\tilde\xi_0 d\tilde{\boldsymbol\xi}d\mathbf{y}\big|.
\end{aligned}
$$

Next, notice that

$$
\begin{aligned}
&\frac{1}{\delta_n^{(d+1)/2}}\mathcal{F}(u_n(\cdot,\cdot,\mathbf{y},\lambda))(\tilde\xi_0/\delta_n,\tilde{\boldsymbol\xi}/\delta_n)=\delta_n^{(d+1)/2}\mathcal{F}(u_n(\delta_n\,\cdot,\delta_n\,\cdot,\mathbf{y},\lambda))(\tilde\xi_0,\tilde{\boldsymbol\xi})\\
&\frac{1}{\delta_n^{(d+1)/2}}\mathcal{F}(v_n(\cdot,\cdot,\mathbf{y}))(\tilde\xi_0/\delta_n,\tilde{\boldsymbol\xi}/\delta_n)=\delta_n^{(d+1)/2}\mathcal{F}(v_n(\delta_n\,\cdot,\delta_n\,\cdot,\mathbf{y}))(\tilde\xi_0,\tilde{\boldsymbol\xi}).
\end{aligned} \qquad (10)
$$

From here, according to the assumptions (i) and (ii) of the theorem, and taking into account that $\Lambda^0=\Lambda_\varepsilon\cup\Lambda^\varepsilon$ and that $\mathrm{meas}(\Lambda_\varepsilon)\to 0$ as $\varepsilon\to 0$, we conclude the theorem.

❑

Having the latter theorem in mind, we can derive information about support of the defect functional from the localization principle as follows.

Lemma 7. *Let $\tilde\mu$ be the defect measure defined in Theorem 6 and assume that the conditions of Theorem 6 hold. Assume that the function $F\in C_0(\mathbf{R}^{d+1}\times B(0,1))$ is such that for some $\alpha>0$*

$$
\sup_{(\mathbf{y},\xi_0,\boldsymbol\xi)\in R^d\times\mathbf{R}^{1+d}}\mathrm{meas}\{\lambda\in K_\lambda:\ |iF\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol\xi)}{|(\xi_0,\boldsymbol\xi)|+1}\right)+F(\mathbf{y},\lambda,0)|\leq\sigma\}\leq\sigma^\alpha. \qquad (11)
$$

Furthermore, assume that

$$
\left(F\left(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol\xi)}{|(\xi_0,\boldsymbol\xi)|+1}\right)\chi_{\Lambda_0(\boldsymbol\xi)}+F(\mathbf{y},\lambda,0)\chi_{\Lambda^0(\boldsymbol\xi)}\right)d\tilde\mu\equiv 0. \qquad (12)
$$

Then,

$$\tilde\mu\equiv 0.$$

Remark 8. It is enough to take any function tending to zero as $\sigma\to 0$ in (11) instead of σ^α.

Proof: Fix an arbitrary $\delta > 0$, and for $\psi \in C_0(\mathbf{R}^d \times K_\lambda \times [-1,1]^{d+1})$ consider the test function

$$\psi\left(\mathbf{y},\lambda,\mathbf{w}\right)\frac{\overline{F(\mathbf{y},\lambda,\mathbf{w}))}}{|F(\mathbf{y},\lambda,\mathbf{w})|^2+\delta}.$$

From (12), we have with the notation from Theorem 6

$$0 = \langle \frac{\psi|F|^2}{|F|^2+\delta},\mu\rangle = \langle \psi,\mu\rangle + \langle \frac{\psi\delta}{|F|^2+\delta},\mu\rangle. \tag{13}$$

We let here $\delta \to 0$ and conclude

$$\langle \psi,\mu\rangle = 0$$

which we intended to prove. It remains to prove that

$$\lim_{\delta\to 0}\langle \frac{\psi\delta}{|F|^2+\delta},\mu\rangle = 0. \tag{14}$$

According to (7) and the bound (6), we have

$$\begin{aligned}&\langle \frac{\psi\delta}{|F|^2+\delta},\mu\rangle \qquad (15)\\ &=\delta\langle \left(\frac{\psi(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|^2+1})}{|F(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|^2+1})|^2+\delta}\chi_{\Lambda_0}(\boldsymbol{\xi}) + \frac{\psi(\mathbf{y},\lambda,0)}{|F(\mathbf{y},\lambda,0)|^2+\delta}\chi_{\Lambda^0}(\boldsymbol{\xi})\right),\tilde{\mu}\rangle.\end{aligned}$$

According to the assumption (11), it is not difficult to see :-)

$$\sup_{\xi_0,\boldsymbol{\xi},\mathbf{y}}\|\frac{\psi(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|^2+1})}{|F(\mathbf{y},\lambda,\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|^2+1})|^2+\delta}\chi_{\Lambda_0}(\boldsymbol{\xi})\|_{L^2(K_\lambda)} \leq c < \infty;$$

$$\sup_{\xi_0,\boldsymbol{\xi},\mathbf{y}}\|\frac{\psi(\mathbf{y},\lambda,0)}{|F(\mathbf{y},\lambda,0)|^2+\delta}\chi_{\Lambda_0}(\boldsymbol{\xi})\|_{L^2(K_\lambda)} \leq c < \infty.$$

From here, (6) and (15), we conclude (14). ❑

3. Existence of traces for quasi-solutions to (1)

In this section, we shall first define quasi-solutions to (1). The notion is introduced in [30] and it is a generalization of the Kruzhkov-type admissibility concept (see e.g. [10, 11, 20]). In a special situation, the quasi-solution is an entropy admissible solution that singles out a physically relevant solutions to the equation (1). The notion of quasi-solution will lead to an appropriate kinetic formulation of the equation under consideration which will enable us to use the defect measures.

Definition 9. A measurable function u defined on $\mathbf{R}^d_+$ is called a quasi-solution to (1) if for every $\lambda \in \mathbf{R}$ the Kruzhkov type entropy equality holds

$$\begin{aligned}&\partial_t|u-\lambda| + \operatorname{div}\left[\operatorname{sgn}(u-\lambda)(\mathfrak{f}(u)-\mathfrak{f}(\lambda)\right]] \qquad (16)\\ &-D^2\cdot[\operatorname{sgn}(u-\lambda)(A(u)-A(\lambda))] = -\gamma(t,\mathbf{x},\lambda),\end{aligned}$$

where $\gamma \in C(\mathbf{R}_\lambda; w\star-\mathcal{M}(\mathbf{R}^d_+))$ we call the quasi-entropy defect measure.

From the latter definition, the following kinetic formulation can be proved.

Theorem 10. *Denote $F = \mathfrak{f}'_\lambda$ and $a = A'_\lambda$. If the function u is a quasi-solution to* (1) *then the function*

$$h(t,\mathbf{x},\lambda) = \operatorname{sgn}(u(t,\mathbf{x}) - \lambda) = -\partial_\lambda |u(t,\mathbf{x}) - \lambda| \tag{17}$$

is a weak solution to the following linear equation:

$$\partial_t h + \operatorname{div}(F(\lambda)h) - D^2 \cdot [a(\lambda)h] = \partial_\lambda \gamma(t,\mathbf{x},\lambda) \tag{18}$$

Proof: It is enough to find derivative of (16) with respect to $\lambda \in \mathbf{R}$ to obtain (18). ❑

Next, we introduce the non-degeneracy conditions.

Definition 11. Denote

$$\mathcal{L}(\lambda,\xi_0,\boldsymbol{\xi}) = \frac{i(\xi_0 + \sum\limits_{k=1}^{d} F_k(\lambda)\xi_k) + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\xi_0,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}.$$

We say that the coefficients of equation (1) satisfy the non-degeneracy conditions if there exists $\alpha > 0$ such that for every $\sigma > 0$ and every finite interval $I \subset \mathbf{R}$

$$\operatorname{esssup}_{\mathbf{x}\in K} \sup_{|\boldsymbol{\xi}|=1} \operatorname{meas}\{\lambda \in I : \ |\mathcal{L}(\lambda,\xi_0,\boldsymbol{\xi})| \leq \sigma\} \leq C(I)\sigma^\alpha, \tag{19}$$

where $C(I)$ is a constant depending only on I.

Such kind of assumptions are standard in the theory of velocity averaging lemmas [21, 31, 32] (see in particular (see also [32, (2.18)-(2.19)])) which is substantially used in the frame of the blow up method [36]. Here, we cannot use known velocity averaging results due to specific form of the transport equation that we are going to obtain after appropriate (blow up) change of variables (see (23)). Let us also remark that instead of σ^α on the right-hand side of (19) we can use some other function tending to zero as $\sigma \to 0$.

The main theorem of the paper is:

Theorem 12. *Assume that the quasi-solution u to* (1) *satisfies for some constant $M > 0$*

$$-M \leq u(t,\mathbf{x}) \leq M, \quad \text{a.e.} \quad (t,\mathbf{x}) \in \mathbf{R}^d_+.$$

Then, if the non-degeneracy conditions (11) *are satisfied, the function u admits the strong trace at $t = 0$ i.e. there exists $u_0 \in L^\infty(\mathbf{R}^d)$ such that for any relatively compact $K \subset\subset \mathbf{R}^d$, it holds*

$$\lim_{t\to 0} \int_K |u(t,\mathbf{x}) - u_0(\mathbf{x})| d\mathbf{x} = 0.$$

Let us first prove that a weak solution to (18) admits a weak trace.

Proposition 13. *Let $h \in L^\infty(\mathbf{R}^d_+ \times \mathbf{R})$ be a distributional solution to* (18)*. Denote*

$$E = \{t \in \mathbf{R}^+ : \ (t,\mathbf{x},\lambda) \text{ is the Lebesgue point of } h(t,\mathbf{x},\lambda) \text{ for a.e. } (\mathbf{x},\lambda) \in \mathbf{R}^d \times \mathbf{R}\}. \tag{20}$$

There exists $h_0 \in L^\infty(\mathbf{R}^{d+1})$, such that

$$h(t,\cdot,\cdot) \rightharpoonup h_0, \ \text{weakly-}\star \text{ in } L^\infty(\mathbf{R}^{d+1}), \ \text{as } t \to 0, \ t \in E.$$

Proof: Since $h \in L^\infty(\mathbf{R}^d_+ \times \mathbf{R})$, the family $\{h(t,\cdot,\cdot)\}_{t\in E}$ is bounded in $L^\infty(\mathbf{R}^{d+1})$. Due to weak-$\star$ precompactness of $L^\infty(\mathbf{R}^{d+1})$, there exists a sequence $\{t_m\}_{m\in\mathbf{N}}$, $t_m \to 0$, as $m \to \infty$, and $h_0 \in L^\infty(\mathbf{R}^{d+1})$, such that

$$h(t_m,\cdot,\cdot) \rightharpoonup h_0(\cdot,\cdot), \text{ weakly-}\star \text{ in } L^\infty(\mathbf{R}^{d+1}), \text{ as } m \to \infty. \tag{21}$$

For $\phi \in C^\infty_c(\mathbf{R}^d)$, $\rho \in C^1_c(\mathbf{R})$, denote

$$I(t) := \int_{\mathbf{R}^{d+1}} h(t,\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda, \quad t \in E.$$

With this notation, (21) means that

$$\lim_{m\to\infty} I(t_m) = \int_{\mathbf{R}^{d+1}} h_0(\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda =: I(0). \tag{22}$$

Now, fix $\tau \in E$ and notice that for the regularization $I_\varepsilon = I \star \omega_\varepsilon$ where ω_ε is the standard convolution kernel, it holds

$$\lim_{\varepsilon\to 0} I_\varepsilon(\tau) = I(\tau).$$

Then, fix $m_0 \in \mathbf{N}$, such that $E \ni t_m \le \tau$, for $m \ge m_0$, and remark that

$$\begin{aligned}
I(\tau) - I(t_m) &= \lim_{\varepsilon\to 0}\int_{t_m}^{\tau} I'_\varepsilon(t)\,dt = \int_{t_m}^{\tau}\int_{\mathbf{R}^{d+1}} \partial_t h(t,\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda\,dt \\
&= \sum_{i=1}^{d}\int_{(t_m,\tau]\times\mathbf{R}^{d+1}} h(t,\mathbf{x},\lambda)F_i(\lambda)\rho(\lambda)\partial_{x_i}\phi(\mathbf{x})\,d\mathbf{x}d\lambda dt \\
&+ \sum_{i,j=1}^{k}\int_{(t_m,\tau]\times\mathbf{R}^{d+1}} h(t,\mathbf{x},\lambda)a_{ij}(\lambda)\rho(\lambda)\partial_{x_i x_j}\phi(\mathbf{x})\,d\mathbf{x}d\lambda dt \\
&- \int_{(t_m,\tau]\times\mathbf{R}^{d+1}} \phi(\mathbf{x})\rho'(\lambda)\,d\gamma(t,\mathbf{x},\lambda).
\end{aligned}$$

Now, since E is the set of full measure, we can choose the sequence (t_m) of numbers from E converging to zero. Now, passing to the limit as $m \to \infty$, and having in mind (22), we obtain

$$\begin{aligned}
I(\tau) - I(0) &= \sum_{i=1}^{d}\int_{(0,\tau]\times\mathbf{R}^{d+1}} h(t,\mathbf{x},\lambda)F_i(\lambda)\rho(\lambda)\partial_{x_i}\phi(\mathbf{x})\,d\mathbf{x}\,d\lambda\,dt \\
&+ \sum_{i,j=1}^{k}\int_{(0,\tau]\times\mathbf{R}^{d+1}} a_{ij}(\lambda)\rho(\lambda)h(t,\mathbf{x},\lambda)\partial_{x_i x_j}\phi(\mathbf{x})\,d\mathbf{x}\,d\lambda\,dt \\
&- \int_{(0,\tau]\times\mathbf{R}^{d+1}} \rho'(\lambda)\phi(\mathbf{x})\,d\gamma(t,\mathbf{x},\lambda) \underset{\tau\to 0}{\longrightarrow} 0.
\end{aligned}$$

Thus, for all $\phi \in C^\infty_c(\mathbf{R}^d)$, $\rho \in C^1_c(\mathbf{R})$,

$$\lim_{\tau\in E,\tau\to 0}\int_{\mathbf{R}^{d+1}} h(\tau,\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\lambda\,d\mathbf{x} = \int_{\mathbf{R}^{d+1}} h_0(\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\lambda\,d\mathbf{x}.$$

Having in mind that $h(\tau,\cdot)$ is bounded for almost every $\tau \in \mathbf{R}$, and $C^\infty_c(\mathbf{R}^{d+1})$ is dense in $L^1(\mathbf{R}^{d+1})$, we complete the proof. ❑

Next, change the variables in (18) in the following way, $t = \frac{\hat{t}}{m}$, $x_1 = y_1 + \frac{\hat{x}_1}{m}, \dots, x_d = y_d + \frac{\hat{x}_d}{m}$, i.e.

$$(t, \mathbf{x}, \lambda) = (\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{m} + \mathbf{y}, \lambda), \tag{23}$$

where $y \in \mathbf{R}^d$ is a fixed vector. We get for $h^m(\hat{t}, \hat{\mathbf{x}}, \mathbf{y}, \lambda) = h(\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{m} + \mathbf{y}, \lambda)$

$$L_{h^m} := \frac{1}{m}\left(\partial_{\hat{t}} h^m + \sum_{k=1}^d \partial_{\hat{x}_k}\left(F_k h^m\right)\right) - \sum_{s,j=1}^d \partial^2_{\hat{x}_s \hat{x}_j}(a_{sj} h^m) = \frac{1}{m^2}\partial_\lambda \gamma^m, \tag{24}$$

$$h^m|_{t=0} = h_0(\varepsilon_m \hat{\mathbf{x}} + \mathbf{y}, \lambda), \tag{25}$$

where the initial conditions are understood in the weak sense. Let us remark that the equality between γ and γ^m is understood in the sense of distributions:

$$\langle \gamma_m, \varphi \rangle = m^{d+1} \int_{\mathbf{R}^d_+} \varphi(m\, t, m\,(\mathbf{x} - \mathbf{y}), \lambda) d\gamma(t, \mathbf{x}, \lambda) \tag{26}$$

for almost every $\lambda \in \mathbf{R}$. If we prove that there exists $\alpha > 0$ such that for any $\rho \in C^1_c(\mathbf{R})$, the sequence

$$\int h(\frac{\hat{t}}{m^\alpha}, \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}, \lambda)\rho(\lambda)d\lambda, \quad m \in \mathbf{N}, \tag{27}$$

converges strongly in $L^1_{loc}(\mathbf{R}^{1+d} \times \mathbf{R}^d)$ along a subsequence, we will obtain that function u admits the trace in the sense of Definition 1. More precisely the following proposition holds.

Proposition 14. *Assume that for every $\rho \in C^1_c(\mathbf{R})$ the sequence given by (27) converges toward $\int h_0(\mathbf{y}, \lambda)\rho(\lambda)d\lambda$ in $L^1_{loc}(\mathbf{R}^d_+ \times \mathbf{R}^d_{\mathbf{y}})$ along a subsequence. Then, the function u admits the strong trace at $t = 0$ and it is equal to h_0.*

Proof: Using the density arguments, we conclude that if the sequence from (27) converges in $L^1_{loc}(\mathbf{R}^d_+ \times \mathbf{R}^d)$ for any $\rho \in C^1_c(\mathbf{R})$, then it will also converge for any $\rho \in L^\infty_c(\mathbf{R})$. From there, we conclude that for any non-negative $\varphi \in C_c(\mathbf{R}^d_+ \times \mathbf{R}^d)$, it holds

$$\lim_{m\to\infty} \int_{\mathbf{R}^d_+} \varphi(\hat{t}, \hat{\mathbf{x}}, \mathbf{y}) | \int_{-M}^M (h(\frac{\hat{t}}{m^\alpha}, \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}, \lambda) - h_0(\mathbf{y}, \lambda))\rho(\lambda)d\lambda | d\mathbf{y}d\hat{\mathbf{x}}d\hat{t} = 0.$$

Introducing the change of variables $\mathbf{z} = \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}$ with respect to $\mathbf{y}$ here, we conclude

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d_+} \varphi(\hat{t}, \hat{\mathbf{x}}, \mathbf{z} - \frac{\hat{\mathbf{x}}}{m^\alpha}) \times \tag{28}$$

$$\times | \int_{-M}^M (h(\frac{\hat{t}}{m^\alpha}, \mathbf{z}, \lambda) - h_0(\mathbf{z} - \frac{\hat{\mathbf{x}}}{m^\alpha}, \lambda)) d\lambda | d\mathbf{y}d\hat{\mathbf{x}}d\hat{t} = 0.$$

Using the definition of the function h (it is a *sign* function; see (17))

$$\int_{-M}^M h^m(\hat{t}, \hat{\mathbf{x}}, \mathbf{y}, \lambda)d\lambda = \int_{-M}^M \operatorname{sign}(\lambda - u(\frac{\hat{t}}{m^\alpha}, \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}))d\lambda = 2u(\frac{\hat{t}}{m^\alpha}, \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}).$$

From here and the continuity in L^1_{loc} for L^1_{loc} functions (h_0 in particular), we conclude from (28)

$$\lim_{m\to\infty}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d_+}\varphi(\hat{t},\hat{\mathbf{x}},\mathbf{z})|(2u(\frac{\hat{t}}{m^\alpha},\mathbf{z})-\int_{-M}^{M}h_0(\mathbf{z},\lambda))d\lambda|d\mathbf{y}d\hat{\mathbf{x}}d\hat{t}=0.$$

Due to arbitrariness of φ, we conclude

$$u(\frac{\hat{t}}{m^\alpha},\mathbf{z})\to\frac{1}{2}\int_{-M}^{M}h_0(\mathbf{z},\lambda))d\lambda,\quad m\to\infty$$

in $L^1_{loc}(\mathbf{R}^d_+)$ along the subsequence from the formulation of the proposition. This means that for almost every $\hat{t}\in R^+$

$$u(\frac{\hat{t}}{m^\alpha},\mathbf{z})\to\frac{1}{2}\int_{-M}^{M}h_0(\mathbf{z},\lambda))d\lambda=u_0(\mathbf{z}),\quad m\to\infty \tag{29}$$

in $L^1_{loc}(\mathbf{R}^d)$. Now, choose $\rho(\lambda)=\lambda\,\chi_{[-M,M]}(\lambda)$ where $\chi_{[-M,M]}(\lambda)$ is the characteristic function of the interval $[-M,M]$. It holds according to Proposition 13

$$\int_{-M}^{M}\lambda h(t,\mathbf{x},\lambda)d\lambda=u^2(t,\mathbf{x})-M^2\overset{\star}{\rightharpoonup}\frac{1}{2}\int_{-M}^{M}\lambda h_0(\mathbf{y},\lambda)d\lambda-M^2\ \text{ in }\ L^\infty(\mathbf{R}^d\times\mathbf{R}^d)$$

as $t\to0$. Since u is bounded and since weak and strong limits coincide, from here and (29) we see that it must be

$$u^2(t,\mathbf{z})\overset{\star}{\rightharpoonup}u_0^2(\mathbf{z})\ \text{ in }\ L^\infty(\mathbf{R}^d_+)\ \text{ as }\ t\to0.$$

Therefore, for $\varphi\in C_c(\mathbf{R}^d)$, it holds

$$\int_{\mathbf{R}^d}(u(t,\mathbf{z})-u_0(\mathbf{z}))^2\varphi(\mathbf{z})d\mathbf{z}=\int_{\mathbf{R}^d}\left(u(t,\mathbf{z})^2-2u(t,\mathbf{z})u_0(\mathbf{z})+u_0(\mathbf{z})^2\right)\varphi(\mathbf{z})d\mathbf{z}\to0$$

as $t\to0$ implying

$$u(t,\cdot)\to\frac{1}{2}\int_{-M}^{M}h_0(\cdot,\lambda))d\lambda,\ t\to0$$

in $L^1_{loc}(\mathbf{R}^d)$. This concludes the proof. ❑

Having the last proposition in mind, we clearly need the following theorem.

Theorem 15. *Assume that the non-degeneracy conditions* (19) *from Definition 11 are satisfied.*

Then, for any $\rho\in C^1_c(\mathbf{R})$, *the sequence* $(\int_{\mathbf{R}}\rho(\lambda)h(\hat{t}/m^{1/(d+1)},\hat{\mathbf{x}}/m^{1/(d+1)}+\mathbf{y},\lambda)d\lambda)$ *is strongly precompact in* $L^1_{loc}(\mathbf{R}^d_+\times\mathbf{R}^d)$.

Proof: Assume in the sequel that $d\geq2$. Take an arbitrary $\varphi\in L^2(\mathbf{R}^{1+d})\cap C^2_c(\mathbf{R}^{1+d})$ and multiply (24) by

$$\varphi_m(\hat{t},\hat{\mathbf{x}})=\frac{1}{m^\beta}\varphi(\frac{(\hat{t},\hat{\mathbf{x}})}{m^{\frac{2\beta}{d+1}}}),\quad\frac{2\beta}{d+1}<1/2. \tag{30}$$

We can rewrite the obtained equation in the following way (equality holds in the weak sense):

$$
\begin{aligned}
&\frac{1}{m}\left(\partial_{\hat t}(\varphi_m\, h^m)+\sum_{k=1}^d \partial_{\hat x_k}\left(F_k(\lambda) h^m\varphi_m\right)\right)-\sum_{s,j=1}^d a_{sj}(\lambda)\partial^2_{\hat x_s \hat x_j}(h^m\varphi_m) \qquad (31)\\
&-\frac{1}{m}\left(\partial_{\hat t}\varphi_m\, h^m+h^m F(\lambda)\cdot\nabla\varphi_m\right)\\
&-2\sum_{k,j=1}^d a_{jk}(\lambda)\partial_{x_j}\left(h_m\partial_{x_k}\varphi_m\right)+h_m\sum_{k,j=1}^d a_{jk}(\lambda)\partial_{x_k x_j}\varphi_m=\frac{1}{m^2}\varphi_m\partial_\lambda\gamma^m.
\end{aligned}
$$

Denote $u_m=\varphi_m\, h^m$ and remark that (u_m) is bounded in $L^2(\mathbf{R}^d_+)$.

Next, recall that

$$
\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle=\sum_{k=1}^d\left(\sum_{j=1}^d \sigma_{kj}(\lambda)\xi_j\right)^2
$$

and thus, in particular, $a_{kj}=\sum\limits_{i=1}^d \sigma_{ki}\sigma_{ij}$. From here, it is easy to see that

$$
\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\eta}\rangle=\sum_{i=1}^d\left(\sum_{k=1}^d \sigma_{ki}(\lambda)\xi_k\right)\left(\sum_{j=1}^d \sigma_{ji}(\lambda)\eta_j\right),\quad \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbf{R}^d. \qquad (32)
$$

Now, fix $\rho\in C^1_c(\mathbf{R}^{d+1})$ and introduce the sequence of functions

$$
v_m(\hat t,\hat{\mathbf{x}},\mathbf{y},\lambda)=\rho(\mathbf{y},\lambda)\overline{\mathcal{A}}_{\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)\frac{1}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}(\omega_m(\cdot,\cdot,\mathbf{y}))(\hat t,\hat{\mathbf{x}}),
$$

for a fixed sequence (ω_m) (defined on $\mathbf{R}^d_+\times\mathbf{R}^d$) weakly-$\star$ converging to zero in $L^\infty\cap L^2(\mathbf{R}^d_+\times\mathbf{R}^d)$ (the sequence will be specified later). In the sequel, to simplify the notations, we shall omit the conjugation sign but we shall imply it whenever we use the Plancherel theorem.

For every fixed m, we test (31) against v_m. We get (we label each line below since we will consider each of it separately):

$$\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, u_m\, i\Big(\mathcal{A}_{\frac{\xi_0}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})}(\omega_m) \tag{33}$$

$$+\sum_{k=1}^d F_k(\lambda)\mathcal{A}_{\frac{\xi_k}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})}(\omega_m)\Big)d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda$$

$$\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} (-\partial_t\varphi_m\, h^m - h^m F(\lambda)\cdot\nabla\varphi_m)\,\rho(\mathbf{y},\lambda)\times \tag{34}$$

$$\overline{\mathcal{A}}_{\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})\frac{1}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}}(\omega_m(\cdot,\cdot,\mathbf{y}))(\hat{t},\hat{\mathbf{x}})d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda$$

$$+\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, u_m \mathcal{A}_{\frac{m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})}(\omega_m)d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda \tag{35}$$

$$+2\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, h^m \sum_{k,j=1}^d a_{jk}(\lambda)\partial_{x_j}\varphi_m\times \tag{36}$$

$$\times\,\partial_{x_k}\mathcal{A}_{\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})\frac{1}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}(\omega_m)d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda$$

$$+\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, h^m\left(\partial_t\varphi_m+\sum_{k,j=1}^d a_{jk}(\lambda)\partial_{x_kx_j}\varphi_m\right)\times \tag{37}$$

$$\times\,\mathcal{A}_{\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})\frac{1}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}(\omega_m)d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda$$

$$=\frac{1}{m^2}\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}}\varphi_m\partial_\lambda v_m(\hat{t},\hat{\mathbf{x}},\lambda,\mathbf{y})d\gamma^m(\hat{t},\hat{\mathbf{x}},\mathbf{y},\lambda). \tag{38}$$

Let us consider line by line in (33)–(38) as $m\to\infty$.

As for (33), denote by μ the defect measure corresponding to the sequences $(u_m)=(\varphi_m h^m)$ and (ω_m). Letting $m\to\infty$ in (33), we get

$$\lim_{m\to\infty}\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, u_m\, i\Big(\mathcal{A}_{\frac{\xi_0}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})}(\omega_m) \tag{39}$$

$$+\sum_{k=1}^d F_k(\lambda)\mathcal{A}_{\frac{\xi_k}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\psi(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1})}(\omega_m)\Big)d\hat{t}d\hat{\mathbf{x}}d\lambda d\mathbf{y}$$

$$=i\langle\rho(\mathbf{y},\lambda)\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+1}\right)(\frac{\xi_0}{|(\xi_0,\boldsymbol{\xi})|+1}+\sum_{k=1}^d F_k(\lambda)\frac{\xi_k}{|(\xi_0,\boldsymbol{\xi})|+1}),\tilde{\mu}(\mathbf{y},\lambda,\xi_0,\boldsymbol{\xi})\rangle.$$

The term (34) clearly tends to zero as $m\to\infty$ sincethe derivative of φ_m tends to zero in $L^2(\mathbf{R}^d_+)$.

It is also easy to handle (35). Notice that

$$\frac{m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}=1-\frac{|(\xi_0,\boldsymbol{\xi})|+1}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

which means that we can rewrite (35) in the form

$$\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, u_m \mathcal{A}_{\left(1-\frac{|(\xi_0,\boldsymbol{\xi})|+1}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right)}(\omega_m) d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda$$

Letting $m\to\infty$ here, we get after taking into account $\beta<\frac{4}{d_1}$ and the change of variables $(\tilde{t},\tilde{\mathbf{x}})=\left(\frac{t}{m^{2\beta/(d+1)}},\frac{\mathbf{x}}{m^{2\beta/(d+1)}}\right)$:

$$\lim_{m\to\infty}\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho\, u_m \mathcal{A}_{\frac{m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\xi_0,\boldsymbol{\xi})|+m\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)}(\omega_m) d\hat{t}d\hat{\mathbf{x}}d\lambda d\mathbf{y} \tag{40}$$
$$=\langle \rho(\lambda,\mathbf{y})\psi(0),\tilde{\mu}(\lambda,\mathbf{y},\xi_0,\boldsymbol{\xi})\rangle.$$

Now, let us handle term (36). Using (32), we can rewrite (36) as:

$$\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho h^m \sum_{k,j=1}^d a_{jk}(\lambda)\partial_{x_j}\varphi_m\, \partial_{x_k}\mathcal{A}_{\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)\frac{1}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}(\omega_m) d\hat{t}d\hat{\mathbf{x}}d\lambda d\mathbf{y}$$

$$=\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho h^m \sum_{s=1}^d \left(\sigma_{js}(\lambda)\partial_{x_j}\varphi_m\right)\times$$
$$\times\left(\mathcal{A}_{\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)\frac{\sum_{k=1}^d\sigma_{sk}(\lambda)\xi_k}{\frac{1}{m}|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}(\omega_m)\right) d\hat{t}d\hat{\mathbf{x}}d\lambda d\mathbf{y}$$

Using (32), we see that it holds

$$\frac{\sum_{k=1}^d\sigma_{sk}(\lambda)\xi_k}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}=\frac{\sum_{k=1}^d\sigma_{sk}(\lambda)\xi_k}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\sum_{s=1}^d(\sum_{k=1}^d\sigma_{sk}(\lambda)\xi_k)^2+\frac{1}{m}},$$

which, together with the Marcinkiewicz multiplier theorem (Theorem 3) readily implies that for any fixed λ

$$\Big\|\frac{1}{\sqrt{m}}\mathcal{A}_{\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right)\frac{\sum_{k=1}^d\sigma_{sk}(\lambda)\xi_k}{\frac{1}{m}|(\xi_0,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\frac{1}{m}}}\Big\|_{L^p\to L^p}\leq C(p)\|\psi\|_{C^d([-1,1]^{d+1})},\ \ p>1, \tag{41}$$

where $C(p)$ is a constant depending on p (but it is independent of m). Next, notice that for $\beta>2/(d+1)$ we have

$$\sqrt{m}\sigma_{js}(\lambda)\partial_{x_j}\varphi_m\to 0 \ \text{ in } \ L^r(\mathbf{R}^d_+) \tag{42}$$

for r large enough.

From (41) and (42), we conclude that term (36) tends to zero as $m\to\infty$.

In the completely same way, we conclude that term (37) tends to zero.

Let us now consider term (38). Remark first that it holds for any symbol ψ depending on λ

$$\partial_\lambda\mathcal{A}_\psi(\omega_m)=\mathcal{A}_{\partial_\lambda\psi}(\omega_m).$$

Now, it is a moment to choose (ω_m). We take for a bounded function ω

$$\omega_m(t,\mathbf{x},\mathbf{y})=\varphi_m(t,\mathbf{x})\omega(\frac{t}{m},\mathbf{y}+\frac{\mathbf{x}}{m}).$$

It holds by the definition of the Fourier multiplier operators (below, $(\tau,\mathbf{z})\in\mathbf{R}\times\mathbf{R}^d$)

$$\begin{aligned}
&\mathcal{A}_\psi\left(\varphi\left(\frac{\cdot}{m^{d/(d+1)}}\right)\omega\left(\frac{(\cdot,\mathbf{y}+\cdot)}{m}\right)\right)(m^{d/(d+1)}\tau,m^{d/(d+1)}\mathbf{z}) \qquad (43)\\
&=\int e^{2\pi i\, m^{d/(d+1)}\,(\tau,\mathbf{z})\cdot(\xi_0,\boldsymbol{\xi})}\psi(\xi_0,\boldsymbol{\xi})\times\\
&\qquad\times\int e^{-2\pi i(t,\mathbf{x})\cdot(\xi_0,\boldsymbol{\xi})}\varphi(\frac{(t,\mathbf{x})}{m^{d/(d+1)}})\omega(\frac{t}{m},\frac{\mathbf{x}}{m}+\mathbf{y})dtd\mathbf{x}d\xi_0 d\boldsymbol{\xi}\\
&=\left(m^{d/(d+1)}(\xi_0,\boldsymbol{\xi})=(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})\right)\\
&=\int\frac{1}{m^d}e^{2\pi i(\tau,\mathbf{z})\cdot(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}\psi\left(\frac{(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}{m^{d/(d+1)}}\right)\times\\
&\qquad\times\int e^{-2\pi i\frac{(t,\mathbf{x})}{m^{d/(d+1)}}\cdot(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}\varphi\left(\frac{t}{m^{d/(d+1)}},\frac{\mathbf{x}}{m^{d/(d+1)}}\right)\omega(\frac{t}{m},\frac{\mathbf{x}}{m}+\mathbf{y})dtd\mathbf{x}d\tilde{\xi}_0 d\tilde{\boldsymbol{\xi}}\\
&=\left(\frac{(t,\mathbf{x})}{m^{d/(d+1)}}=(\tilde{t},\tilde{\mathbf{x}})\right)\\
&=\int e^{2\pi i(\tau,\mathbf{z})\cdot(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}\psi\left(\frac{(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}{m^{d/(d+1)}}\right)\times\\
&\qquad\times\int e^{-2\pi i(\tilde{t},\tilde{\mathbf{x}})\cdot(\tilde{\xi}_0,\tilde{\boldsymbol{\xi}})}\varphi(\tilde{t},\tilde{\mathbf{x}})\omega(\frac{\tilde{t}}{m^{1/(d+1)}},\mathbf{y}+\frac{\tilde{\mathbf{x}}}{m^{1/(d+1)}})d\tilde{t}d\tilde{\mathbf{x}}d\tilde{\xi}_0 d\tilde{\boldsymbol{\xi}}\\
&=\mathcal{A}_{\psi(\cdot/m^{d/(d+1)})}(\varphi(\cdot,\cdot)\omega(\cdot/m^{1/(d+1)},\mathbf{y}+\cdot/m^{1/(d+1)}))(\tau,\mathbf{z}).
\end{aligned}$$

Therefore, introducing the change of variables

$$(\tau,\mathbf{z})=\frac{(\hat{t},\hat{\mathbf{x}})}{m^{2\beta/(d+1)}}$$

in (38) we get

$$\begin{aligned}
&\frac{1}{m^2}\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}}\varphi_m\partial_\lambda v_m(\hat{t},\hat{\mathbf{x}},\lambda,\mathbf{y})d\gamma^m(\hat{t},\hat{\mathbf{x}},\mathbf{y},\lambda) \qquad (44)\\
&=\frac{1}{m^2}\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}}\varphi(\tau,\mathbf{z})\partial_\lambda(\rho(\lambda,\mathbf{y})\times\\
&\qquad\times\mathcal{A}_{\psi_m(\lambda,\boldsymbol{\xi})}\left(\varphi w(\frac{\cdot}{m^{1/(d+1)}},\frac{\cdot}{m^{1/(d+1)}}+\mathbf{y})\right)(\tau,\mathbf{z}))d\lambda d\tilde{\gamma}^m(\tau,\mathbf{z},\lambda,\mathbf{y}),
\end{aligned}$$

where

$$\psi_m=\psi\left(\frac{(\xi_0,\boldsymbol{\xi})}{|(\xi_0,\boldsymbol{\xi})|+m^{1/(d+1)}\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}\right)\frac{m^{2-\frac{1}{d+1}}}{|(\xi_0,\boldsymbol{\xi})|+m^{-1/(d+1)}\langle a\boldsymbol{\xi},\boldsymbol{\xi}\rangle+1}$$

and

$$\int\phi(\tau,\mathbf{z},\mathbf{y},\lambda)d\tilde{\gamma}^m=\int_{R^{1+d}_+\times\mathbf{R}^{d+1}}m\phi(m^{1/(d+1)}t,m^{1/(d+1)}(\mathbf{x}-\mathbf{y}),\lambda,\mathbf{y})d\mathbf{y}d\gamma(t,\mathbf{x},\lambda).$$

Let us first prove that

$$\frac{1}{m^{1/(d+1)}}\tilde{\gamma}^m \to 0 \ \text{ in } \ \mathcal{M}_{loc}(\mathbf{R}^d_+ \times \mathbf{R}^{d+1}). \tag{45}$$

To this end, denote by $B_r = \{|\mathbf{x}| \leq r;\ \mathbf{x} \in \mathbf{R}^d\}$ the ball centred at zero with the diameter $r > 0$ in the L^∞-norm. Denote by $\kappa(t, \mathbf{x})$ the indicator function of the set $(0, r) \times B(0, r)$ and take $\phi \in C_c(\mathbf{R}^{d+1})$ arbitrary supported in $B_L \times [-L, L]$. We have

$$\begin{aligned}&\frac{1}{m^{1/(d+1)}}\int_{(0,r)\times B_r \times \mathbf{R}^{d+1}} \phi(\lambda, \mathbf{y})d\tilde{\gamma}^m \\ &= \frac{1}{m^{1/(d+1)}}\int_{R^d_+ \times \mathbf{R}^{d+1}} m\phi(\lambda, \mathbf{y})\kappa(m^{1/(d+1)}t, m^{1/(d+1)}(\mathbf{x}-\mathbf{y}))d\mathbf{y}d\gamma(t, \mathbf{x}, \lambda).\end{aligned}$$

Introducing the change of variables, $\tau = m^{1/(d+1)}t$, $\mathbf{z} = m^{1/(d+1)}(\mathbf{x}-\mathbf{y})$. We get:

$$\begin{aligned}&\frac{1}{m^{1/(d+1)}}\int_{(0,r)\times B_r \times \mathbf{R}^{d+1}} \phi(\lambda, \mathbf{y})d\tilde{\gamma}^m \\ &= \frac{1}{m^{1/(d+1)}}\int_{\mathbf{R}^d_+}\int_{\mathbf{R}^{d+1}} \kappa(\tau, \mathbf{z})\phi(\lambda, \mathbf{y})d\mathbf{y}d\gamma(\frac{t}{m^{1/(d+1)}}, \frac{\mathbf{x}}{m^{1/(d+1)}}, \lambda) \to 0,\ m \to 0,\end{aligned}$$

which proves (45). Remark that the change of variables used here, since it is linear, applies also on measures (i.e. we can consider it as a function $\gamma = \gamma(t, \mathbf{x}, \lambda)dtd\mathbf{x}d\lambda$).

Next, using the Marcinkiewicz multiplier theorem (Theorem 3), we see that $\frac{\psi_m}{m^{2-\frac{1}{d+1}}}$ and $\frac{\partial_\lambda \psi_m}{m^{2-\frac{1}{d+1}}}$ define for every fixed $\lambda \in \mathbf{R}$ the continuous multiplier operators

$$\begin{aligned}\mathcal{A}_{\frac{\psi_m}{m^{2-\frac{1}{d+1}}}} &= \frac{1}{m^{2-\frac{1}{d+1}}}\mathcal{A}_{\psi_m} : L^p(\mathbf{R}^d_+) \to W^{1,p}(\mathbf{R}^d) \\ \mathcal{A}_{\frac{\partial_\lambda\psi_m}{m^{2-\frac{1}{d+1}}}} &= \frac{1}{m^{2-\frac{1}{d+1}}}\mathcal{A}_{\partial_\lambda\psi_m} : L^p(\mathbf{R}^d_+) \to W^{1,p}(\mathbf{R}^d),\end{aligned} \tag{46}$$

where $p > 1$ is arbitrary. From here and the Sobolev embedding theorems, we conclude that

$$\partial_\lambda(\rho(\lambda, \mathbf{y})\mathcal{A}_{\psi_m}(\varphi\,\omega(\frac{\tau}{m^{1/(d+1)}}, \mathbf{y} + \frac{\mathbf{z}}{m^{1/(d+1)}})))$$

remains bounded in $L^p(\mathbf{R}^{d+1}; C(\mathbf{R}^d_+))$, $p > 1$ is arbitrary. Therefore, from here, (44) and (45), we conclude that (38) also tends to zero as $m \to \infty$.

From the previous considerations, we conclude that after taking limit as $m \to \infty$ in (33)–(38), we reach to

$$\int_{\mathbf{R}^{d+1}\times\mathbf{R}^{d+1}} \psi(\boldsymbol{\xi})\rho(\mathbf{y}, \lambda)\Bigg((\frac{\xi_0}{|(\xi_0, \boldsymbol{\xi})|+1} + \sum_{k=1}^d F_k(\lambda)\frac{\xi_0}{|(\xi_0, \boldsymbol{\xi})|+1})\chi_{\Lambda_0(\boldsymbol{\xi})} + \chi_{\Lambda^0(\boldsymbol{\xi})}\Bigg)d\tilde{\mu}(\mathbf{y}, \lambda, \xi_0, \boldsymbol{\xi}) = 0.$$

From here, using Lemma 7 and having in mind conditions (19), we conclude that the defect measure $\tilde{\mu}$ is zero and therefore $\mu \equiv 0$ as well. Inserting $\psi \equiv 1$ in (3) we obtain

$$\lim_{m\to\infty}\int_{\mathbf{R}^d_+\times\mathbf{R}^{d+1}} \rho(\lambda, \mathbf{y})u_m(\hat{t}, \hat{\mathbf{x}}, \mathbf{y}, \lambda)\omega_m(\hat{t}, \hat{\mathbf{x}})d\hat{t}d\hat{\mathbf{x}}d\mathbf{y}d\lambda = 0. \tag{47}$$

Now, we choose in (47)

$$\omega_m(\hat{t},\hat{\mathbf{x}})=\varphi_m(\hat{t},\hat{\mathbf{x}})\omega(\frac{\hat{t}}{m},\frac{\hat{\mathbf{x}}}{m}+\mathbf{y})$$
$$=\int_{\mathbf{R}}\rho(\mathbf{y},\lambda)\varphi_m(\hat{t},\hat{\mathbf{x}})\left(h^m(\hat{t},\hat{\mathbf{x}},\mathbf{y},\lambda)-h^0(\mathbf{y},\lambda)\right)d\lambda \rightharpoonup 0 \quad \text{in} \quad L^\infty(\mathbf{R}^d_+\times\mathbf{R}^{d+1}).$$

If we recall that $h^m(\hat{t},\hat{\mathbf{x}},\lambda)=h(\frac{\hat{t}}{m},\mathbf{y}+\frac{\hat{\mathbf{x}}}{m},\lambda)$ and introduce the change of variables $(\frac{\hat{t}}{m^{d/(d+1)}},\frac{\hat{\mathbf{x}}}{m^{d/(d+1)}})=(\tau,\mathbf{z})$ in (47), we get:

$$\int\limits_{\mathbf{R}^d_+\times\mathbf{R}^d}\varphi^2(\tau,\mathbf{z})\left(\int_{\mathbf{R}}\rho(\lambda,\mathbf{y})(h(\frac{\tau}{m^{1/(d+1)}},\mathbf{y}+\frac{\mathbf{z}}{m^{1/(d+1)}},\lambda)-h_0(\mathbf{y},\lambda))d\lambda\right)^2 d\tau d\mathbf{z}d\mathbf{y}\to 0$$

as $m\to 0$. The proof is concluded. ❑

Now, we have the proof of the main theorem.

Proof of Theorem 12 The proof directly follows from Theorem 15 and Proposition 14. □

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