VELOCITY AVERAGING FOR DIFFUSIVE TRANSPORT EQUATIONS WITH DISCONTINUOUS FLUX

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Abstract. We consider a diffusive transport equation with discontinuous flux and prove the velocity averaging result under non-degeneracy conditions. In order to achieve the result, we introduce a new variant of micro-local defect functionals which are able to "recognise" changes of the type of the equation. As a corollary, we show the existence of a solution for the Cauchy problem for nonlinear degenerate parabolic equation with discontinuous flux. We also show existence of strong traces at $t = 0$ for so-called quasi-solutions to degenerate parabolic equations under non-degeneracy conditions on the diffusion term.

1. Introduction

In [44, Theorem C] a result on velocity averaging for diffusive transport equations has been stated, but the proof of the theorem cannot be found neither in that paper or in later contributions (we shall provide a more detailed insight later in the introduction). The aim of the paper is to precisely proof [44, Theorem C] in the L^q-setting, $q > 2$, and to generalise the result on equations with discontinuous coefficients.

To be more precise, we aim to prove a velocity averaging result for a diffusive transport equation with discontinuous flux meaning that for the sequence (u_n) of solutions to the sequence of equations of the form

$$\begin{aligned} \operatorname{div}_{\mathbf{x}}\big(f(\mathbf{x},\lambda)u_n(\mathbf{x},\lambda)\big) &= \operatorname{div}_{\mathbf{x}}\big(\operatorname{div}_{\mathbf{x}}\left(a(\lambda)u_n(\mathbf{x},\lambda)\right)\big) \\ &\qquad + \partial_\lambda G_n(\mathbf{x},\lambda) + \operatorname{div}_{\mathbf{x}} P_n(\mathbf{x},\lambda) \qquad \text{in } \mathcal{D}'(\mathbb{R}^{d+1}) \,, \end{aligned} \tag{1}$$

for every $\rho \in C^1_c(\mathbb{R})$, the sequence $\big(\int_{\mathbb{R}} \rho(\lambda) u_n(\mathbf{x},\lambda)\, d\lambda\big)$ is strongly precompact in $L^1_{loc}(\mathbb{R}^d)$ (i.e. it lies in a compact subset of $L^1_{loc}(\mathbb{R}^d)$).

Equation (1) has two main components. The transport part

$$\operatorname{div}_{\mathbf{x}}(f(\mathbf{x},\lambda)u_n(\mathbf{x},\lambda))$$

and the diffusive part

$$\operatorname{div}_{\mathbf{x}}\big(\operatorname{div}_{\mathbf{x}}\left(a(\lambda)u_n(\mathbf{x},\lambda)\right)\big) = \operatorname{div}_{\mathbf{x}}\big(a(\lambda)\nabla_{\mathbf{x}} u_n(\mathbf{x},\lambda)\big) \,,$$

where $u_n(\mathbf{x},\lambda)$ is unknown, $a(\lambda) \in \mathbb{R}^{d\times d}$ is the diffusion matrix, $f(\mathbf{x},\lambda)$ is the flux, $\mathbf{x} \in \mathbb{R}^d$ is the space (and time) variable and λ is called the velocity variable, but it can be considered as a parameter. For the sake of generality, and simplicity of the exposition, we compressed the space-time variable into a single variable $\mathbf{x}$, while still our main intention is to study evolution equations (see Remark 6). In the literature, velocity variable λ is often denoted by v. The form of the remaining source terms in (1) is motivated by the kinetic formulation for degenerate parabolic equations, as can be seen in Section 5 and Section 6.

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The transport component $\operatorname{div}_{\mathbf{x}}(f(\mathbf{x},\lambda)u_n(\mathbf{x},\lambda))$ is a generalisation of the usual kinetic transport term $\langle v \,|\, \nabla_{\mathbf{x}} h(\mathbf{x},v)\rangle$, i.e. the equation

$$\partial_t h + \langle v \,|\, \nabla_{\mathbf{x}} h\rangle = \operatorname{div}_{\mathbf{x}} \partial_v^{\boldsymbol{\kappa}} g, \quad (t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^d, \quad v\in\mathbb{R}^d, \quad g \in \mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)\,, \ \boldsymbol{\kappa}\in\mathbb{N}^d\,,$$

for which the velocity averaging results was proved in [1]. Independently of [1], the corresponding results were discovered in [33] and further extended in [32]. The mentioned results were given in the L^2-setting. In [21], one can find the first L^p, $p>1$, velocity averaging result obtained using the approach of multiplier operators (see e.g. [34]). The optimal result in the sense of the L^p-integrability of (u_n) has been achieved in [11, 51], while an L^2 velocity averaging result for pseudo-differential operators can be found in [27].

Such a type of result appeared to be very useful and it was a substantial part of the proof of existence of the weak solution to the Boltzmann equation [20] as well as the regularity of admissible solutions to scalar conservation laws [44]. In [44], one can also find the first result concerning the velocity averaging for the transport equations with the flux of the form $f=f(\lambda)$, $f\in \mathrm{C}(\mathbb{R};\mathbb{R}^d)$, under the non-degeneracy conditions which essentially mean that for any $\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}$, the mapping

$$\lambda \mapsto \langle f(\lambda) \,|\, \boldsymbol{\xi}\rangle \tag{2}$$

is possibly zero only on a negligible set.

As for the non-hyperbolic situation ($a\neq 0$), the velocity averaging results for ultra-parabolic equations are proven in [40], while for the degenerate parabolic equations, i.e. the ones in which a changes rank for different λ, by our best knowledge, the only results can be found in [29, 44, 55] for the homogeneous flux f and diffusion matrix a (i.e. both independent of $\mathbf{x}$) in the L^p-settig for any $p>1$. Let us note here again that in [44] details of the proof are not provided (see [44, Theorem C]) since the authors conjectured that the proof could be accomplished by following the method from [32]. However, such a method is actually applied in [55] and the authors needed an additional assumption (see [55, (2.20)]) to finalise the arguments (see more precise explanation below).

Let us now briefly explain a main idea of the technique from [29, 44, 55]. Since both flux and diffusion matrix are independent of $\mathbf{x}$, this enables a separation of coefficients and unknown functions by means of the Fourier transform. Indeed, if f is independent of $\mathbf{x}$, by applying the Fourier transform to (1) with respect to $\mathbf{x}$ one sees that

$$\hat{u}_n(\boldsymbol{\xi},\lambda) = \frac{\sigma^2|\boldsymbol{\xi}|^2\hat{u}_n + i\langle \hat{P}_n \,|\, \boldsymbol{\xi}\rangle + \partial_\lambda \hat{G}_n}{\sigma^2|\boldsymbol{\xi}|^2 + i\langle f \,|\, \boldsymbol{\xi}\rangle + 2\pi\langle a\,\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle}\,, \tag{3}$$

where we denoted by $\boldsymbol{\xi}$ the dual variable (the definition of the Fourier transform used here can found in Notation below). In the case when $a\equiv 0$, informally speaking, from (3):

- by controlling the term $\hat{u}_n$ on the right-hand side of the latter expression by the constant σ;
- by integrating by parts with respect to λ to remove the derivative from the functions $\hat{G}_n$;
- by employing the non-degeneracy conditions (2);

one can draw appropriate conclusions on the sequence (u_n).

The generalisation on the situation when $a\neq 0$ is not straightforward. First we need to assume that

$$(\forall \boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}) \qquad \operatorname{meas}\bigl\{\lambda\in K\subset\subset\mathbb{R} : \langle f(\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\bigr\} = 0\,, \tag{4}$$

which are the non-degeneracy conditions corresponding to (1) (with the flux independent of $\mathbf{x}$).

However, since the integration by parts with respect to λ (which is the second step in the procedure above) affects the non-negativity of the matrix a, it seems that additional assumptions on a are needed in order to conclude about the strong compactness of the velocity averages.

In particular, such a result can be found in [55], which is aimed to the regularity properties of the velocity averages (more precisely, $\mathrm{W}^{s,r}$-regularity, $s > 0$, $r \geq 1$). In the essence of the proofs is the method described above (separation of the solution u from coefficients) together with the so-called *truncation property* [55, Definition 2.1] (see [55, Lemma 2.3]) under a variant of assumption (4) and an assumption on behavior of the λ-derivative of the symbol $\mathcal{L}(\boldsymbol{\xi}, \lambda) = i\langle f(\lambda)\,|\,\boldsymbol{\xi}\rangle + 2\pi\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle$ of equation (1) on layers in $\boldsymbol{\xi}$-space defined by the Littlewood–Palley decomposition. In [29], the results are repeated in the stochastic setting.

We also mention results from [28, 30] where one can find velocity averaging results for degenerate parabolic equations obtained as a kinetic reformulation of the porous media equation.

Before stating our main result, let us first fix the notation used in the paper.

Notation. Throughout the paper we denote by $\langle\cdot\,|\,\cdot\rangle$ the complex Euclidean scalar product on $\mathbb{C}^d$, which we take to be antilinear in the second argument. However, in our situations we shall mainly work on $\mathbb{R}^d$. By $|\cdot|$ we denote the corresponding norm of vectors, while the same notation is used for the 2-norm for matrices. For a matrix A, by A^T we denote its transpose. For the complex conjugate of a complex number z we use $\bar{z}$.

By $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ we write points (vectors) in $\mathbb{R}^d$, while by $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_d)$ we denote the dual variables in the sense of the Fourier transform (if t occurs, then we use τ for the dual variable). The Fourier transform we define by $\hat{u}(\boldsymbol{\xi}) = \mathcal{F}u(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i\langle\boldsymbol{\xi}\,|\,\mathbf{x}\rangle} u(\mathbf{x})\,d\mathbf{x}$, and its inverse by $(u)^{\vee}(\mathbf{x}) = \bar{\mathcal{F}}u(\mathbf{x}) = \int_{\mathbb{R}^d} e^{2\pi i\langle\boldsymbol{\xi}\,|\,\mathbf{x}\rangle} u(\boldsymbol{\xi})\,d\mathbf{x}$, while the Fourier multiplier operator by $\mathcal{A}_\psi u = (\psi\hat{u})^{\vee}$. If $\mathcal{A}_\psi$ is bounded on $\mathrm{L}^p(\mathbb{R}^d)$ we call it the L^p-Fourier multiplier operator and ψ the L^p-Fourier multiplier. We will often have that ψ depends (besides $\boldsymbol{\xi}$) on λ which is always considered as a parameter.

For a Lebesgue measurable subset $A \subseteq \mathbb{R}^d$ we denote by $Cl\,A$, cA, $\mathrm{meas}(A)$ and χ_A the closure of A, the complement of A, the Lebesgue measure of A, and the characteristic function over A, respectively. The open (closed) ball in $\mathbb{R}^d$ centered at point $\mathbf{x}$ with radius $r > 0$ we will denote by $B(\mathbf{x}, r)$ ($B[\mathbf{x}, r]$), the unit sphere in $\mathbb{R}^d$ by S^{d-1}, and in Section 5 we will use the shorthand $\mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d$. The signum function is denoted by sgn.

For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we denote by $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ its length and by $\partial^{\boldsymbol{\alpha}} = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}$ partial derivatives.

By $\mathrm{L}^p_{loc}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open and $p \in [1, \infty]$, we denote the Fréchet space of functions that are contained in $\mathrm{L}^p(\Omega')$ for any compactly contained set Ω' in Ω ($\Omega' \subset\subset \Omega$), and analogously for Sobolev spaces $\mathrm{W}^{s,p}_{loc}(\Omega)$, $s \in \mathbb{R}$, $p \in [1, \infty]$. $\mathrm{C}_c(X)$ stands for the space of compactly supported continuous functions on a locally compact space X. If X is compact then $\mathrm{C}_c(X) = \mathrm{C}(X)$. For the space of Lipschitz functions we use $\mathrm{C}^{0,1}(X)$. Any dual product is denoted by $\langle\cdot,\cdot\rangle$, which we take to be linear in both arguments. $\mathcal{L}(X)$ stands for the space of bounded linear operators on a normed space X.

When applicable, functions defined on a subset of $\mathbb{R}^d$ shall often be identified by their extensions by zero to the whole space.

In order to introduce the main results of the paper, we need the following assumptions on (u_n) and the coefficients appearing in (1):

Assumptions

a) (u_n) is uniformly compactly supported on open $\Omega \times S \subset\subset \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}_\lambda$, $d \geq 2$, and weakly (weakly-$\star$ for $q = \infty$) converges to zero in $\mathrm{L}^q(\mathbb{R}^d \times \mathbb{R})$ for some $q \in (2, \infty]$;
b) $a = \sigma^T\sigma$, where $\sigma \in \mathrm{C}^{0,1}(S; \mathbb{R}^{d\times d})$;
c) $f \in \mathrm{L}^p(\Omega \times S; \mathbb{R}^d)$ for some $p > \frac{q}{q-1}$ ($p > 1$ if $q = \infty$), and for any compact $K \subseteq S$ it holds

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\ \sup_{\boldsymbol{\xi}\in\mathrm{S}^{d-1}}\ \mathrm{meas}\big\{\lambda \in K : \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\big\} = 0\,; \tag{5}$$

d) $G_n \to 0$ strongly in $\mathrm{L}^{r_0}_{loc}(\mathbb{R}_\lambda; \mathrm{W}^{-1/2,r_0}_{loc}(\mathbb{R}^d_{\mathbf{x}}))$ for some $r_0 \in (1, \infty)$;

e) $P_n = (P_1^n, \dots, P_d^n) \to 0$ strongly in $\mathrm{L}^{p_0}_{loc}(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}_\lambda; \mathbb{R}^d)$ for some $p_0 \in (1, \infty)$.

Our main result is the following velocity averaging result for (1).

Theorem 1. *Let $d \geq 2$ and let (u_n) satisfies (a) and the sequence of equations* (1) *whose coefficients satisfy conditions (b)–(e).*

Then there exists a subsequence $(u_{n'})$ such that for any $\rho \in \mathrm{C}_c(S)$,

$$\int_S \rho(\lambda) u_{n'}(\mathbf{x}, \lambda)\, d\lambda \longrightarrow 0 \quad \textit{strongly in } \mathrm{L}^1_{loc}(\mathbb{R}^d)\,. \tag{6}$$

The theorem above generalises the compactness results of [44, 55] to the case of the flux discontinuous with respect to the space variable, while the diffusion matrix remains homogeneous, i.e. dependent only on λ. Moreover, the non-degeneracy condition (5) can be seen as a natural generalisation of (4) to the heterogeneous setting. The heterogeneity prevents us of using the above explaind method based on the Fourier transform, thus in our proof we follow the approach of [26, 40], which is elaborated below. However, we are not able to obtain the result for (u_n) bounded in L^p if $p \leq 2$, as achieved in [44, 55].

Assumption (b) from the above, on the diffusion matrix a, can be relaxed (see Remark 23), and the proofs remain essentially the same. The following corollary holds.

Corollary 2. *Assume that (a), (c), (d), (e) from the above are satisfied, and*

$\tilde{\mathrm{b}}$) *the mapping $\lambda \mapsto a(\lambda) = \sigma^T(\lambda)\sigma(\lambda) \in \mathbb{R}^{d\times d}$ is such that for almost every $\lambda_0 \in S \subset\subset \mathbb{R}$ there exists $\varepsilon > 0$ such that $\sigma \in \mathrm{C}^{0,1}((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon); \mathbb{R}^{d\times d})$.*

Then (6) *holds.*

Moreover, under stronger assumptions on (G_n) we have the following result.

Corollary 3. *If we replace (b) and (d) by*

b') *$a \in \mathrm{C}^{0,1}(S; \mathbb{R}^{d\times d})$ is such that, for every $\lambda \in S$, $a(\lambda)$ is a symmetric and positive semi-definite matrix;*

d') *$G_n \to 0$ strongly in $\mathrm{L}^{r_0}_{loc}(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}_\lambda)$ for some $r_0 \in (1, \infty)$;*

the statement of Theorem 1 still holds.

Compering the result of Corollary 2, when applied to the homogeneous setting ($f = f(\lambda)$), to [44, Theorem C], one can see that the former does not reveal completely the latter, where only smoothness and positive semi-definiteness of a is required, i.e. (b') instead of ($\tilde{\mathrm{b}}$). Nevertheless, conditions (b) and ($\tilde{\mathrm{b}}$) still cover many interesting cases of the degenerate diffusion effects. Let us illustrate this on the following example.

Example 4. a) *It is clear that all matrix functions of the form $a(\lambda) = Q(\lambda)^T \Lambda(\lambda) Q(\lambda)$ satisfy condition (b), where, for any $\lambda \in S$, $Q(\lambda)$ is orthogonal and $\Lambda(\lambda)$ is positive-definite and diagonal, and $Q, \sqrt{\Lambda} \in \mathrm{C}^{0,1}(S; \mathbb{R}^{d\times d})$. Indeed, in this case we can take $\sigma(\lambda) = \sqrt{\Lambda(\lambda)}Q(\lambda)$.*

For instance,

$$a(\lambda) = \left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right)\begin{bmatrix}0 & 0\\ 0 & \lambda^2+1\end{bmatrix}\left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right) = \begin{bmatrix}1 & -\lambda\\ -\lambda & \lambda^2\end{bmatrix}, \tag{7}$$

is of the above form. Therefore, situations where the kernel of $a(\lambda)$ depends on λ are allowed, which overcomes the results of [40, 49] *for ultra-parabolic equations.*

b) *Assumption (b) trivially implies (b'), while*

$$a(\lambda) = \begin{pmatrix}0 & 0\\ 0 & |\lambda|\end{pmatrix} \tag{8}$$

is a simple example which illustrates that the converse does not hold. Indeed, a satisfies (b'), but $\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{|\lambda|} \end{pmatrix}$ *is not Lipschitz continuous around zero, which implies that a matrix* σ *such that condition (b) is satisfied does not exist.*

Since this matrix is singular on the set of zero Lebesgue measure, we can apply Corollary 2 and still obtain the result. However, one can easily generalise (8) *to the case where the singular set of* $\sqrt{a}$ *is of positive measure. For example, take a Cantor set* C *on* $[0,1]$ *of a non-zero measure (so-called fat Cantor set) and on each connected component* (α,β) *of* $[0,1]\setminus C$ *define* $g(\lambda)=\frac{\beta-\alpha}{2}-|\lambda-\frac{\alpha+\beta}{2}|$ *(another possibility could be* $g(\lambda)=(\beta-\lambda)(\lambda-\alpha)$*). Then*

$$\begin{pmatrix} 0 & 0 \\ 0 & g(\lambda) \end{pmatrix} \tag{9}$$

satisfies (b'), but does not satisfy neither (b) nor $(\tilde{b})$*. Thus, for this matrix only Corollary 3 is applicable among our results.*

Remark 5. In Example 4(a) we have seen that smoothness of eigenvectors of $a(\lambda)$ (i.e. smoothness of $Q(\lambda)$) could help in fulfilling condition (b). Let us recall some known results in this direction ([38, II.6.1-3]):

(1) If $a(\lambda)$ is symmetric and analytic then both eigenvectors and eigenvalues are analytic functions;
(2) If $a(\lambda)$ is symmetric and C^1, then eigenvalues are C^1-functions, while eigenvectors need not to be even continuous.

Of course, not even item (1), with addition of positive semi-definiteness, is sufficient to ensure (b) since we require, in principle, Lipschitz continuity of $\sqrt{a}$.

Remark 6. Since we are particularly interested in the parabolic case, we refer to (1) as a degenerate parabolic equation with discontinuous flux, although it is not necessarily of the parabolic type. More precisely, in the application to the Cauchy problem for nonlinear degenerate parabolic equation with discontinuous flux (see Section 5) we shall have $f(t,\mathbf{x},\lambda)=\begin{bmatrix} 1 \\ \tilde{f}(t,\mathbf{x},\lambda) \end{bmatrix}$ and $a(\lambda)=\begin{bmatrix} 0 & 0 \\ 0 & \tilde{a}(\lambda) \end{bmatrix}$. In order to have that a and f satisfy assumptions (b) and (c) it is sufficient to have $\tilde{f}\in L^p(\Omega_{t,\mathbf{x}}\times S;\mathbb{R}^d)$, $\tilde{a}=\tilde{\sigma}^T\tilde{\sigma}$, where $\tilde{\sigma}\in C^{0,1}(S;\mathbb{R}^{d\times d})$, and for any $K\subset\subset S$

$$\operatorname*{ess\,sup}_{(t,\mathbf{x})\in\Omega}\ \sup_{(\tau,\boldsymbol{\xi})\in S^d} \operatorname{meas}\big\{\lambda\in K:\ \tau+\langle\tilde{f}(t,\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle=\langle\tilde{a}(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle=0\big\}=0\,.$$

We shall now briefly explain principles of our approach.

Since we cannot separate the unknown function u_n from the coefficients in (1), here we use variants of micro-local defect measures (or H-measures) introduced in now seminal papers by P. Gerard [26] and L. Tartar [56]. Besides the velocity averaging results [26, 40], the H-measures and similar tools found applications on existence of traces and solutions to nonlinear evolution equations [3, 36, 47], generalisation of compensated compactness results to equations with variable coefficients [26, 56], applications in the control theory [19, 43], explicit formulae and bounds in homogenisation [6, 57], etc.

Moreover, it initiated variety of different generalisations to the original micro-local defect measures which we call here micro-local defect functionals. We mention parabolic and ultra-parabolic variants of the H-measures [7, 50], H-measures as duals of Bochner spaces [40], H-distributions [5, 9, 41, 45], micro-local compactness forms [52], one-scale H-measures [4, 58] etc.

Let us recall the first variant of H-measures [56] (introduced at the same time as the micro-local defect measures [26]).

Theorem 7. *If (u_n) is a sequence in $\mathrm{L}^2_{loc}(\Omega;\mathbb{R}^r)$, $\Omega \subseteq \mathbb{R}^d$, such that $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{loc}(\Omega;\mathbb{R}^r)$, then there exist a subsequence $(u_{n'}) \subset (u_n)$ and a positive complex Radon measure $\mu = \{\mu^{jk}\}_{j,k=1,\dots,r}$ on $\Omega \times \mathrm{S}^{d-1}$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ it holds*

$$\lim_{n'\to\infty} \int_\Omega (\varphi_1 u^j_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\bar\psi(\frac{\cdot}{|\cdot|})}(\varphi_2 u^k_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu^{jk}, \varphi_1 \bar\varphi_2 \psi\rangle$$
$$= \int_{\Omega\times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x})} \psi(\boldsymbol{\xi}) d\mu^{jk}(\mathbf{x},\boldsymbol{\xi}) ,$$

where $\mathcal{A}_{\bar\psi(\frac{\cdot}{|\cdot|})}$ is the Fourier multiplier operator with the symbol $\bar\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$.

The measure μ is called the H-measures and, with respect to the dual variable $\boldsymbol{\xi}$, it is defined on the sphere (since $\boldsymbol{\xi}/|\boldsymbol{\xi}| \in \mathrm{S}^{d-1}$).

It has been proved (see [6]) that applying H-measures on differential relations where the ratio of the highest orders of derivatives in each variable is not the same might lead to unsatisfactory results. This is due to the projection $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}/|\boldsymbol{\xi}|$, since scalings in all variables are the same. We can change the scaling and put, for example, $\frac{\boldsymbol{\xi}}{|(\xi_1,\dots,\xi_k)|+|(\xi_{k+1},\dots,\xi_d)|^2}$ instead of $\boldsymbol{\xi}/|\boldsymbol{\xi}|$, but such H-measure will be able to *see* e.g. first order derivatives with respect to $(x_1,\dots,x_k)$ and second order derivatives with respect to $(x_{k+1},\dots,x_d)$ (for a parabolic variant, see [7]). In other words, no change of the highest order of the equation is permitted. For instance, this means that the matrix $a(\lambda)$ in equation (1) must have the rank and the kernel (locally) independent of λ (see also [50]) otherwise, we cannot use the existing theory of the micro-local defect functionals (except in special situations [36]).

This represents a significant confinement since many challenging mathematical questions, especially from a view-point of modeling, involve equations that change type. In particular, we have in mind degenerate parabolic equations which describe wide range of phenomena containing the combined effects of nonlinear convection and degenerate diffusion and which have the form

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},u) = D^2_{\mathbf{x}} \cdot A(u) , \tag{10}$$

where the matrix A is such that the mapping $\mathbb{R} \ni \lambda \mapsto \langle A(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle$ is non-decreasing, i.e. that the diffusion matrix $A'(\lambda)$ is merely non-negative definite. To this end, let us mention [24], where one of the first results on the case of degenerate parabolic equations was given (to be more precise, an ultra-parabolic equation was considered there).

Let us remark that in the case when the coefficients in (10) are regular, the theory of existence and uniqueness for appropriate Cauchy problems is well-established (see e.g. [17, 18, 31]). Nevertheless, concrete applications such as flow in porous media very often occur in highly heterogeneous environment causing rather rough coefficients in (10) (e.g. during CO_2 sequestration process [46]). Furthermore, even in a simplified situation in which the diffusion is neglected such as a road traffic with variable number of lanes [12], the Buckley-Leverett equation in a layered porous medium [2, 37], and sedimentation applications [14, 13, 22, 23], the flux is as a rule discontinuous.

However, due to obvious technical obstacles, most of the previous literature was dedicated either to homogeneous degenerate parabolic equations or to equations where the flux and diffusion are regular functions (e.g. [10, 15, 16, 17, 60]). We mention here [36] where (10) was considered with the flux $\mathfrak{f}(t,\mathbf{x},\lambda)$ (merely) continuous with respect to λ and belonging to L^p, $p > 2$, with respect to $\mathbf{x} \in \mathbb{R}^d$. Here, we are able to improve the result by relaxing the assumption of p to $p > 1$. More precise explanation can be found in Section 5.

A similar situation is regarding existence of traces. The trace of a function u at $t = 0$ is a function $u_0 \in \mathrm{L}^1_{loc}(\mathbb{R}^d)$ such that

$$u(t,\cdot) \to u_0(\cdot) \ \text{ as } \ t \to 0 \ \text{ in } \ \mathrm{L}^1_{loc}(\mathbb{R}^d) .$$

One can find several results in the hyperbolic setting [47, 48, 59] while we have no results for degenerate parabolic equations of form (44). Special situations were considered in [3, 39]. In [39], one has the scalar diffusion matrix $a(\lambda) = \tilde{a}(\lambda)I$ where I is the unitary matrix and $\tilde{a} \in \mathrm{C}^1(\mathbb{R})$ is a non-negative function. In [3], we considered the situation with ultra-parabolic matrices. In both cases, assumptions were imposed so that the essential problem of λ–changing degeneracy directions does not appear. To be more descriptive, we note that the matrix a given in Example 4 is not covered by the results from [3, 39]. In the current contribution we shall thus provide the first result regarding existence of strong traces in the case when the diffusion matrix degenerates in directions which depend on λ. Moreover, we allow that the flux depends explicitly on $\mathbf{x}$ and it can even be discontinuous.

As we shall see, our tool is robust enough to capture cases of quite rough fluxes and degenerate diffusion at the same time (see Theorem 28).

We overcome this situation by considering multiplier operators with symbols of the form

$$\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\right) , \quad \psi \in \mathrm{C}(\mathbb{R}^d) , \tag{11}$$

where the matrix a represents the diffusion matrix in the degenerate parabolic equation (1).

The paper is organised as follows.

In Section 2 we study symbols of the form (11), which shall be often used for the Fourier multiplier operators, and show two important results concerning their continuity (see Lemma 9), while Section 3 is devoted to the construction of adaptive micro-local defect functionals.

In Section 4, we use the results of sections 2 and 3 to prove the main result of the paper, Theorem 1.

In Section 5, as an application of the velocity averaging result, we show existence of a weak solution to the Cauchy problem of the degenerate advection-diffusion equation with discontinuous flux. The strategy of the proof is to reduce the degenerate parabolic equation (10) to its kinetic counterpart of the form (below, $f = \partial_\lambda \mathfrak{f}$ and $a = A'$):

$$\begin{aligned}\partial_t h(t,\mathbf{x},\lambda) &+ \mathrm{div}_{\mathbf{x}}(f(t,\mathbf{x},\lambda)h(t,\mathbf{x},\lambda))\\ &= \mathrm{div}_{\mathbf{x}}\big(\mathrm{div}_{\mathbf{x}}\,(a(\lambda)h(t,\mathbf{x},\lambda))\big) + \partial_\lambda G(t,\mathbf{x},\lambda) + \mathrm{div} P(t,\mathbf{x},\lambda) ,\end{aligned}$$

and then to use the velocity averaging results.

In Section 6, we provide another application of the velocity averaging result by proving that any bounded quasi-solution to (44) (see Definition 25) admits the strong trace at $t = 0$ under the non-degeneracy condition:

$$\sup_{\boldsymbol{\xi}\in \mathrm{S}^{d-1}} \mathrm{meas}\big\{\lambda \in K : \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\big\} = 0 . \tag{12}$$

2. Results on Fourier integral operators

Let $a : S \to \mathbb{R}^{d\times d}$, $S \subseteq \mathbb{R}$, be a Borel measurable matrix function such that for a.e. $\lambda \in S$ matrix $a(\lambda)$ is symmetric and positive semi-definite, i.e. $a(\lambda)^T = a(\lambda)$ and $\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle \geq 0$, $\boldsymbol{\xi} \in \mathbb{R}^d$. Further on, we define

$$\pi_P(\boldsymbol{\xi},\lambda) := \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} , \quad (\boldsymbol{\xi},\lambda) \in \mathbb{R}^d\backslash\{0\} \times S . \tag{13}$$

As $a(\lambda)$ is positive semi-definite, we have $\pi_P(\mathbb{R}^d\backslash\{0\} \times S) \subseteq B[0,1] \setminus \{0\}$, where $B[0,1]$ denotes the unit closed ball in $\mathbb{R}^d$. Moreover, it is not difficult to show that (for a.e. $\lambda \in S$)

$$Cl\,\pi_P(\mathbb{R}^d\backslash\{0\},\lambda) = \left\{\begin{array}{lcl} \mathrm{S}^{d-1} & : & a(\lambda) = 0 \\ B[0,1] & : & a(\lambda) \neq 0 \end{array}\right. , \tag{14}$$

where $Cl\,A$ denotes the closure of $A \subseteq \mathbb{R}^d$.

If $a(\lambda) = 0$, $\pi_P(\cdot,\lambda)$ is the projection of $\mathbb{R}^d \setminus \{0\}$ to the unit sphere along the rays through the origin. In general $\pi_P(\cdot,\lambda)$ is not a projection since $a(\lambda) \neq 0$ implies $\pi_P(\pi_P(\cdot,\lambda),\lambda) \neq \pi_P(\cdot,\lambda)$. However, for simplicity, in the text we shall often address π_P as a projection.

In this paper, we are interested in symbols of Fourier multipliers of the form

$$\boldsymbol{\xi} \mapsto \bar{\psi}(\pi_P(\boldsymbol{\xi},\lambda),\lambda)\,,$$

where $\lambda \in S$ is fixed, $\psi \in \mathrm{L}^\infty(S;\mathrm{C}(B[0,1]))$, and π_P is as above. Here $\bar{z}$ denotes the complex conjugate of complex number z.

Of course, $\psi \in \mathrm{L}^\infty(S;\mathrm{C}(B[0,1]))$ is sufficient to have that the Fourier multiplier operator is bounded on $\mathrm{L}^2(\mathbb{R}^d)$, with the norm independent on λ. However, we shall need such a result on an arbitrary L^p, for which we need some additional regularity of ψ with respect to $\boldsymbol{\xi}$. More precisely, we shall first obtain that for a.e. λ and for any $p \in (1,\infty)$ operator $\mathcal{A}_{\bar{\psi}(\pi_P(\cdot,\lambda),\lambda)}$ is bounded on $\mathrm{L}^p(\mathbb{R}^d)$, with the norm independent of λ (Lemma 9). Finally, we show that commutators of the Fourier multiplier operators and operators of multiplication map weakly converging sequences to strongly converging in a certain sense (Corollary 15).

In order to prove the L^p boundedness, we use the following corollary of the Marcinkiewicz multiplier theorem [34, Corollary 5.2.5]:

Theorem 8. *Suppose that $\psi \in \mathrm{C}^d(\mathbb{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\})$ is a bounded function such that for some constant $C > 0$ it holds*

$$|\boldsymbol{\xi}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leq C, \quad \boldsymbol{\xi} \in \mathbb{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\} \tag{15}$$

for every multi-index $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$ such that $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq d$. Then ψ is an L^p-multiplier for any $p \in (1,\infty)$, and the operator norm of $\mathcal{A}_\psi$ equals $C_{d,p}C$, where $C_{d,p}$ depends only on p and d.

Before proceeding with the verification of the assumptions of the previous theorem, let us recall some well known results from matrix analysis and at the same time fix our notations.

As $a(\lambda)$ is a non-negative definite symmetric matrix of order d, there exist orthogonal matrix $Q(\lambda)$ and diagonal matrix $\Lambda(\lambda) = \mathrm{diag}(\kappa_1(\lambda),\kappa_2(\lambda),\dots,\kappa_d(\lambda))$, containing (non-negative) eigenvalues of $a(\lambda)$, such that the following eigendecomposition holds:

$$a(\lambda) = Q(\lambda)^T \Lambda(\lambda) Q(\lambda)\,. \tag{16}$$

Furthermore, for $\sigma(\lambda) := \sqrt{\Lambda(\lambda)}Q(\lambda)$, where $\sqrt{\Lambda(\lambda)} = \mathrm{diag}(\sqrt{\kappa_1(\lambda)},\sqrt{\kappa_2(\lambda)},\dots,\sqrt{\kappa_d(\lambda)})$, we thus have

$$a(\lambda) = \sigma(\lambda)^T \sigma(\lambda)\,. \tag{17}$$

On the other hand, if (17) holds for a given by (16), then σ is necessarily of the form

$$\sigma(\lambda) = \widetilde{Q}(\lambda)\sqrt{\Lambda(\lambda)}Q(\lambda)\,, \tag{18}$$

where $\widetilde{Q}(\lambda)$ is an ortogonal matrix.

It is important to notice that

$$\pi_P\big(Q(\lambda)^T\boldsymbol{\xi},\lambda\big) = \frac{Q(\lambda)^T\boldsymbol{\xi}}{|Q(\lambda)^T\boldsymbol{\xi}| + \langle Q(\lambda)a(\lambda)Q(\lambda)^T\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle} = \frac{Q(\lambda)^T\boldsymbol{\xi}}{|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j(\lambda)\xi_j^2}\,,$$

where we have used that $Q(\lambda)^T$ preserves the length of vectors. Hence, with the orthogonal change of variables we will manage to reduce the problem to the case of diagonal matrix a.

Lemma 9. *Let $a : S \to \mathbb{R}^{d\times d}$, $S \subseteq \mathbb{R}$ open, be a Borel measurable matrix function such that for a.e. $\lambda \in S$ matrix $a(\lambda)$ is symmetric and positive semi-definite, and let $\psi \in \mathrm{L}^\infty(S;\mathrm{C}^d(B[0,1]))$.*

Then for a.e. $\lambda \in S$ and any $p \in (1,\infty)$, function $\bar{\psi}(\pi_P(\cdot,\lambda),\lambda)$, where π_P is given by (13), is an L^p-Fourier multiplier and the L^p-norm of the corresponding Fourier multiplier operator is independent of λ.

Proof: Since the space of L^p-Fourier multipliers is invariant under orthogonal change of variables [34, Proposition 2.5.14] (see also Lemma 12 below) and the corresponding norms coincide, applying $\boldsymbol{\xi} \mapsto Q(\lambda)^T\boldsymbol{\xi}$, where $Q(\lambda)$ is given in (16), it is sufficient to study $\boldsymbol{\xi} \mapsto \psi\big(\pi_P(Q(\lambda)^T\,\cdot, \lambda), \lambda\big)$. We shall apply the Marcinkiewicz multiplier theorem (Theorem 8) on this function.

Since $\pi_P(\mathbb{R}^d\backslash\{0\} \times S) \subseteq B[0,1]$, for all $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, $|\boldsymbol{\alpha}| \le d$, functions

$$(\boldsymbol{\xi}, \lambda) \mapsto (\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}}\bar{\psi})\big(\pi_P(Q(\lambda)^T\boldsymbol{\xi}, \lambda), \lambda\big)$$

are bounded on $\mathbb{R}^d\backslash\{0\} \times S$. Therefore, by the generalised chain rule formula (known as the Faá di Bruno formula; see e.g. [35]) it is enough to infer that (15) is satisfied for each component of $\pi_P(Q(\lambda)^T\cdot, \lambda)$, with constant C independent of λ.

Furthermore, since the Riesz transform of order 1 satisfies (15), $Q(\lambda)$ is orthogonal and

$$\big(\pi_P(Q(\lambda)^T\boldsymbol{\xi}, \lambda)\big)_j = \frac{(Q(\lambda)^T\xi)_j}{|\boldsymbol{\xi}| + \langle\Lambda(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} = \frac{(Q(\lambda)^T\xi)_j}{|\boldsymbol{\xi}|}\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}| + \langle\Lambda(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\,,$$

by the Leibniz rule it is sufficient to check (15) for

$$\boldsymbol{\xi} \mapsto \frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}| + \langle\Lambda(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\,.$$

The claim follows by Lemma 10 below. ❑

The proof of the following lemma we leave for the Appendix.

Lemma 10. *For any $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_d) \in [0,\infty)^d$, $m \in \{1, 2, \ldots, d\}$, $s \in [0,\infty)$, and $p \in (1,\infty)$, functions f^s and g^s, where $f, g : \mathbb{R}^d \to \mathbb{R}$ are given by*

$$f(\boldsymbol{\xi}) = \frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2} \qquad \text{and} \qquad g(\boldsymbol{\xi}) = \frac{\kappa_m\xi_m^2}{|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2}\,,$$

and s is the exponent, are L^p-Fourier multipliers and the norm of the corresponding Fourier multiplier operators depends only on d, s and p, i.e. it is independent of $\boldsymbol{\kappa}$. Moreover, f^s and g^s satisfy the Marcinkiewicz condition (15) *with constant C independent of $\boldsymbol{\kappa}$.*

Our next goal is to study the Fourier multiplier operator associated to the symbol $\boldsymbol{\xi} \mapsto \partial_\lambda\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}$. For a smooth a we have

$$\partial_\lambda\frac{1}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} = \frac{-\langle a'(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}{\big(|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle\big)^2} = \psi(\pi_P(\boldsymbol{\xi}, \lambda), \lambda)\,, \tag{19}$$

where $\psi(\boldsymbol{\xi}, \lambda) = -\langle a'(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle$. Thus, by Lemma 9, the Fourier multiplier operator is L^p-bounded, $p \in (1,\infty)$, uniformly in λ, if a' exists (almost everywhere) and it is bounded. However, we need that this operator has a smoothing property.

Let us additionally assume that σ given by (17) is Lipschitz continuous. Since

$$\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = |\sigma(\lambda)\boldsymbol{\xi}|^2 = \sum_{j=1}^d \big(\sigma(\lambda)\boldsymbol{\xi}\big)_j^2\,,$$

we have (for almost every $\lambda \in S$)

$$\langle a'(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = \frac{d}{d\lambda}\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 2\sum_{j=1}^d \big(\sigma(\lambda)\boldsymbol{\xi}\big)_j\big(\sigma'(\lambda)\boldsymbol{\xi}\big)_j\,.$$

Thus, symbol (19) can be rewritten as

$$-2\sum_{j=1}^{d}\frac{1}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}\frac{\big(\sigma(\lambda)\boldsymbol{\xi}\big)_j}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}\frac{\big(\sigma'(\lambda)\boldsymbol{\xi}\big)_j}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\,, \tag{20}$$

and the term $\frac{1}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}$ will provide a smoothing property of the half derivative.

Lemma 11. *In addition to the assumptions in Lemma 9, assume that there exists a Lipschitz continuous matrix function* $\sigma : S \to \mathbb{R}^{d\times d}$ *such that* (17) *holds. Then, for a.e.* $\lambda \in S$ *and any* $p \in (1,\infty)$ *the operator* $\mathcal{A}_{\partial_\lambda \frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{W}^{\frac{1}{2},p}(\mathbb{R}^d)$ *is bounded uniformly with respect to* $\lambda \in S$.

Proof: Since a is a Lipschitz map, a' exists almost everywhere and it is bounded. Thus, $\boldsymbol{\xi} \mapsto -\langle a'(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle$ satisfies assumptions of Lemma 9, and by (19), for any $p \in (1,\infty)$, operator

$$\mathcal{A}_{\partial_\lambda \frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)$$

is uniformly bounded in λ.

To prove that $\mathcal{A}_{\partial_\lambda \frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}$ possesses a smoothing property, we need to prove that derivatives (with respect to $\mathbf{x}$) of the operator are $\mathrm{L}^p \to \mathrm{L}^p$ bounded uniformly in λ:

$$\partial_{x_k}^{\frac{1}{2}}\mathcal{A}_{\partial_\lambda \frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d), \quad k = 1,\dots,d\,. \tag{21}$$

The symbol of the latter operator is

$$\frac{-(2\pi i\xi_k)^{\frac{1}{2}}\langle a'(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}{(|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle)^2}\,,$$

which by (20) can be rewritten as

$$2\sum_{j=1}^{d}\frac{-(2\pi i\xi_k)^{\frac{1}{2}}}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}\frac{\big(\sigma(\lambda)\boldsymbol{\xi}\big)_j}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}\frac{\big(\sigma'(\lambda)\boldsymbol{\xi}\big)_j}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\,.$$

The space of L^p-Fourier multipliers is an algebra [34, Proposition 2.5.13], hence we can study each factor separately.

Since σ' is bounded, by Lemma 9 for a.e. $\lambda \in S$ and any $p \in (1,\infty)$

$$\boldsymbol{\xi} \mapsto \frac{\big(\sigma'(\lambda)\boldsymbol{\xi}\big)_j}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}$$

is an L^p-multipliers with the norm independent of λ. The same holds for

$$\boldsymbol{\xi} \mapsto \frac{-(2\pi i\xi_k)^{\frac{1}{2}}}{\sqrt{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}$$

by Lema 10 (applied on $\sqrt{f}$).

Furthermore, we have $\sigma(\lambda) = \widetilde{Q}(\lambda)\sqrt{\Lambda(\lambda)}Q(\lambda)$ (see (18)). Thus, as the space of L^p-Fourier multipliers is invariant under orthogonal change of variables [34, Proposition 2.5.14] (see also Lemma 12 below) and the corresponding norms coincide, applying $\boldsymbol{\xi} \mapsto Q(\lambda)^T\boldsymbol{\xi}$ and using $|\widetilde{Q}(\lambda)| = 1$, it is left to study

$$\boldsymbol{\xi} \mapsto \frac{\sqrt{\kappa_j(\lambda)}\xi_j}{\sqrt{|\boldsymbol{\xi}|+\sum_{l=1}^d \kappa_l(\lambda)\xi_l^2}}\,.$$

Finally, by Lemma 10 above (applied on $\sqrt{g}$) we have that this mapping is an L^p multiplier with the L^p-norm of the corresponding Fourier multiplier operator independent of functions κ_l, and thus of λ. ❑

It is by now a classical result that if we have a symbol of an L^p-multiplier, then the composition of the symbol with an orthogonal matrix is also a symbol of an L^p-multiplier with the same operator norm (see Proposition 2.5.14 in [34]). Now we will show something very similar when we have a regular change of variables. The result is well known but we include it here for completeness.

Lemma 12. *Let $\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$. If there exists a regular real constant matrix M and $p \in (1,\infty)$ such that $\psi(M^{-1}\cdot)$ is an L^p-multiplier, then ψ is also an L^p-multiplier and $\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} = \|\mathcal{A}_{\psi(M^{-1}\cdot)}\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))}$.*

Proof: Let us denote by $A := \|\mathcal{A}_{\psi(M^{-1}\cdot)}\|_{\mathcal{L}(\mathrm{L}^p)}$ the operator norm, and by $J := |\det M| > 0$ the Jacobian.

Take $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and for an arbitrary $u \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, consider the following:

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(\mathbf{x})\overline{\mathcal{A}_\psi(u)(\mathbf{x})}\,d\mathbf{x} &= \int_{\mathbb{R}^d} \widehat{\varphi}(\boldsymbol{\xi})\overline{\psi(\boldsymbol{\xi})\widehat{u}(\boldsymbol{\xi})}\,d\boldsymbol{\xi}\\ &= J^{-1}\int_{\mathbb{R}^d} \widehat{\varphi}(M^{-1}\boldsymbol{\eta})\overline{\psi(M^{-1}\boldsymbol{\eta})\widehat{u}(M^{-1}\boldsymbol{\eta})}\,d\boldsymbol{\eta}\,,\end{aligned}$$

where we have used Plancherel's theorem in the first equality and the regular change of variables $\boldsymbol{\eta} = M\boldsymbol{\xi}$ in the second one. Furthermore, we have

$$\begin{aligned}\widehat{\varphi}(M^{-1}\boldsymbol{\eta}) &= \int_{\mathbb{R}^d} e^{-2\pi i\mathbf{x}\cdot M^{-1}\boldsymbol{\eta}}\varphi(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^d} e^{-2\pi i M^{-T}\mathbf{x}\cdot\boldsymbol{\eta}}\varphi(\mathbf{x})\,d\mathbf{x}\\ &= J\int_{\mathbb{R}^d} e^{-2\pi i\mathbf{y}\cdot\boldsymbol{\eta}}\varphi(M^T\mathbf{y})\,d\mathbf{y} = \widehat{\varphi(M^T\cdot)}(\boldsymbol{\eta})\,J\,,\end{aligned}$$

where we have used the change of variables $\mathbf{y} = M^{-T}\mathbf{x}$ in the third equality. After applying Plancherel's theorem once more, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \varphi(\mathbf{x})\overline{\mathcal{A}_\psi(u)(\mathbf{x})}\,d\mathbf{x}\right| &= J^{-1}\left|\int_{\mathbb{R}^d} \widehat{\varphi}(M^{-1}\boldsymbol{\eta})\overline{\psi(M^{-1}\boldsymbol{\eta})\widehat{u}(M^{-1}\boldsymbol{\eta})}\,d\boldsymbol{\eta}\right|\\ &= J\left|\int_{\mathbb{R}^d} \widehat{\varphi(M^T\cdot)}(\boldsymbol{\eta})\overline{\psi(M^{-1}\boldsymbol{\eta})\widehat{u(M^T\cdot)}(\boldsymbol{\eta})}\,d\boldsymbol{\eta}\right|\\ &= J\left|\int_{\mathbb{R}^d} \varphi(M^T\mathbf{y})\overline{\mathcal{A}_{\psi(M^{-1}\cdot)}(u(M^T\cdot))(\mathbf{y})}\,d\mathbf{y}\right|\\ &\le JA\|\varphi(M^T\cdot)\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}\|u(M^T\cdot)\|_{\mathrm{L}^p(\mathbb{R}^d)}\\ &\le A\|\varphi\|_{\mathrm{L}^{p'}(\mathbb{R}^d)}\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}\,,\end{aligned}$$

where we have used the Hölder inequality $(1/p + 1/p' = 1)$ and the boundedness of $\mathcal{A}_{\psi(M^{-1}\cdot)}$ in $\mathrm{L}^p(\mathbb{R}^d)$, while the last inequality follows by the fact that the composition with M^T scales the L^p norm of the function by factor $1/\sqrt[p]{|\det M|}$.

From here we conclude that $\mathcal{A}_\psi(u)$ is a continuous linear functional defined on a dense subset of $\mathrm{L}^{p'}(\mathbb{R}^d)$, thus we can uniquely extend it, by the density argument, to a linear functional on the whole $\mathrm{L}^{p'}(\mathbb{R}^d)$, implying that $\mathcal{A}_\psi(u) \in \mathrm{L}^p(\mathbb{R}^d)$ with the following bound:

$$\|\mathcal{A}_\psi(u)\|_{\mathrm{L}^p(\mathbb{R}^d)} \le A\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}\,.$$

The lemma follows for an arbitrary $u \in \mathrm{L}^p(\mathbb{R}^d)$ once we again use the same density argument as above. ❑

In the remaining part of the section we study commutators of Fourier multipliers and operator of multiplications.

In this section we will need a variant of the First commutation lemma which is given in [8, Lemma 1] (see also Remark 2 in the mentioned reference).

Theorem 13. *Let (v_n) be a bounded, uniformly compactly supported sequence in $\mathrm{L}^\infty(\mathbb{R}^d)$, converging to 0 in the sense of distributions, and let $\psi \in \mathrm{C}^d(\mathbb{R}^d\backslash\{0\}) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ be an L^p-multiplier for any $p \in (1,\infty)$ and satisfies*

$$\lim_{|\boldsymbol{\xi}|\to\infty} \sup_{|\mathbf{h}|\leq 1} |\psi(\boldsymbol{\xi}+\mathbf{h}) - \psi(\boldsymbol{\xi})| = 0 \; . \tag{22}$$

Then for any $b \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $r \in (1,\infty)$ the following holds:

$$b\mathcal{A}_\psi(v_n) - \mathcal{A}_\psi(bv_n) \longrightarrow 0 \quad \textit{strongly in} \quad \mathrm{L}^r_{loc}(\mathbb{R}^d) \; .$$

In the following lemma we show that symbols of the form (11) satisfy condition (22).

Lemma 14. *Under assumptions of Lemma 9, for a.e. $\lambda \in S$ function $\bar\psi(\pi_P(\cdot,\lambda),\lambda)$ satisfies (22).*

Proof: Since $\bar\psi(\cdot,\lambda)$ is uniformly continuous on $B[0,1]$, it is sufficient to prove that vector valued function $\pi_P(\cdot,\lambda)$ satisfies (22). Moreover, since (22) is invariant under orthogonal change of coordinates, it is sufficient to study $\pi_P(Q(\lambda)^T\cdot,\lambda)$, where orthogonal matrix $Q(\lambda)$ is given by (16).

For an arbitrary $|\mathbf{h}|\leq 1$ let us estimate $|\pi_P\big(Q(\lambda)^T\boldsymbol{\xi},\lambda\big) - \pi_P\big(Q(\lambda)^T(\boldsymbol{\xi}+\mathbf{h}),\lambda\big)|$. To make the calculations easier to read, we omit the fixed parameter λ. Thus, we have

$$\begin{aligned}
&\Big|\frac{Q^T\boldsymbol{\xi}}{|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} - \frac{Q^T(\boldsymbol{\xi}+\mathbf{h})}{|\boldsymbol{\xi}+\mathbf{h}| + \langle \Lambda(\boldsymbol{\xi}+\mathbf{h})\,|\,\boldsymbol{\xi}+\mathbf{h}\rangle}\Big| \\
&\quad\leq \frac{|Q^T\boldsymbol{\xi} - Q^T(\boldsymbol{\xi}+\mathbf{h})|}{|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} + |Q^T(\boldsymbol{\xi}+\mathbf{h})|\Big|\frac{1}{|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} - \frac{1}{|\boldsymbol{\xi}+\mathbf{h}| + \langle \Lambda(\boldsymbol{\xi}+\mathbf{h})\,|\,\boldsymbol{\xi}+\mathbf{h}\rangle}\Big| \\
&\quad\leq \frac{1}{|\boldsymbol{\xi}|} + |\boldsymbol{\xi}+\mathbf{h}|\frac{\big||\boldsymbol{\xi}+\mathbf{h}| - |\boldsymbol{\xi}|\big| + \big|\langle \Lambda(\boldsymbol{\xi}+\mathbf{h})\,|\,\boldsymbol{\xi}+\mathbf{h}\rangle - \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle\big|}{\big(|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle\big)\big(|\boldsymbol{\xi}+\mathbf{h}| + \langle \Lambda(\boldsymbol{\xi}+\mathbf{h})\,|\,\boldsymbol{\xi}+\mathbf{h}\rangle\big)} \\
&\quad\leq \frac{2}{|\boldsymbol{\xi}|} + \frac{\langle \Lambda\mathbf{h}\,|\,\mathbf{h}\rangle + 2|\langle \Lambda\boldsymbol{\xi}\,|\,\mathbf{h}\rangle|}{|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} \\
&\quad\leq \frac{2+|\Lambda|}{|\boldsymbol{\xi}|} + \frac{2\sqrt{|\Lambda|}\sqrt{\langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}{|\boldsymbol{\xi}| + \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} \, ,
\end{aligned}$$

where in the last line we have used the Cauchy–Bunjakovskij–Schwartz inequality for semi-definite scalar product $(\boldsymbol{\xi},\boldsymbol{\eta}) \mapsto \langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\eta}\rangle$.

If $\langle \Lambda(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0$, by the computations above

$$\big|\pi_P\big(Q(\lambda)^T\boldsymbol{\xi},\lambda\big) - \pi_P\big(Q(\lambda)^T(\boldsymbol{\xi}+\mathbf{h}),\lambda\big)\big| \leq \frac{2+|\Lambda|}{|\boldsymbol{\xi}|} \, ,$$

while for $\langle \Lambda(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle \neq 0$ we have

$$\big|\pi_P\big(Q(\lambda)^T\boldsymbol{\xi},\lambda\big) - \pi_P\big(Q(\lambda)^T(\boldsymbol{\xi}+\mathbf{h}),\lambda\big)\big| \leq \frac{2+|\Lambda|}{|\boldsymbol{\xi}|} + \frac{2\sqrt{|\Lambda|}}{\sqrt{\langle \Lambda\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} \, .$$

In both cases the limit as $|\boldsymbol{\xi}|$ goes to infinity of the difference is zero, implying the claim. ❑

By the previous lemma and Lemma 9, all assumptions of Theorem 13 are satisfied, hence the following corollary holds.

Corollary 15. *Let (v_n) be a bounded, uniformly compactly supported sequence in $\mathrm{L}^\infty(\mathbb{R}^d)$, converging to 0 in the sense of distributions, and let ψ and a be as in Lemma 9.*

Then for any $b \in \mathrm{L}^\infty(\mathbb{R}^d)$, $r \in (1,\infty)$ and a.e. $\lambda \in S$ the following holds:

$$b\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n) - \mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(bv_n) \longrightarrow 0 \quad \text{strongly in} \quad \mathrm{L}^r_{loc}(\mathbb{R}^d)\,.$$

3. Adaptive micro-local defect functionals

In what follows we will have an uniformly compactly supported sequence $(u_n(\mathbf{x},\lambda))$. It means that there exists a bounded open subset $\Omega\times S \subseteq \mathbb{R}^{d+1}$ of finite Lebesgue measure such that supports of all functions u_n are contained in it. Let us take one such $\Omega\times S$ and fix it.

Now, we need to introduce the space on which we shall define the appropriate micro-local defect functional. The space will be adapted to the considered equation (1). For $p\in(1,\infty)$ we define

$$\widetilde{W}^p_\Pi(\Omega,S) = \left\{ \sum_{j=1}^k \varphi_j(\mathbf{x})\psi_j(\boldsymbol{\xi},\lambda) : \, k\in\mathbb{N}\,,\ \varphi_j\in\mathrm{L}^p(\Omega)\,,\, \psi_j\in\mathrm{C}_c(B[0,1]\times S)\,,\, j=1,\dots,k \right\},$$

where $B[0,1]$ is the unit closed ball in $\mathbb{R}^d$. We denote for $\Psi=\Psi(\mathbf{x},\boldsymbol{\xi},\lambda)\in\widetilde{W}^p_\Pi(\Omega,S)$

$$\|\Psi\|_{W^p_\Pi} = \left(\int_\Omega \left[\sup_{\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}} \left(\int_S |\Psi\big(\mathbf{x},\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)|^2\,d\lambda\right)^{1/2}\right]^p d\mathbf{x}\right)^{1/p}, \tag{23}$$

where π_P is given by (13). Due to (14), for $a\not\equiv 0$, this map represents a norm on $\widetilde{W}^p_\Pi(\Omega,S)$. On the other hand, for $a\equiv 0$ it is only a seminorm, so one needs to consider the quotient space by its kernel, or, equivalently, just replace $B[0,1]$ by S^{d-1} in the definition of $\widetilde{W}^p_\Pi(\Omega,S)$.

Finally, we introduce the space $W^p_\Pi(\Omega,S)$ as the completion of $\widetilde{W}^p_\Pi(\Omega,S)$ with respect to the norm $\|\cdot\|_{W^p_\Pi}$. It is easy to see that $W^p_\Pi(\Omega,S)$ coincides with the Bochner space $\mathrm{L}^p(\Omega;X)$, where X is the completion of $\mathrm{C}(B[0,1];\mathrm{L}^2(S))$ equipped with the norm

$$\mathrm{C}(B[0,1];\mathrm{L}^2(S)) \ni \psi \mapsto \sup_{\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}} \left(\int_S |\psi\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)|^2\,d\lambda\right)^{1/2}.$$

Moreover, the space $\mathrm{C}_0(\Omega\times B[0,1]\times S)$, equipped by the standard (supremum) topology, is for any $p\in(1,\infty)$ dense in $W^p_\Pi(\Omega,S)$, so continuous linear functionals on $W^p_\Pi(\Omega,S)$ are in fact bounded Radon measures on $\Omega\times B[0,1]\times S$.

In the following theorem we construct one such functional which will play an important role in the proof of the velocity averaging result. The construction is based on the Banach-Alaoglu-Bourbaki theorem, which applies on $W^p_\Pi(\Omega,S)$ as it is clearly a (separable) Banach space.

Theorem 16. *Let $(u_n(\mathbf{x},\lambda))$ be an uniformly compactly supported sequence on $\Omega\times S\subset\subset\mathbb{R}^d\times\mathbb{R}$ weakly converging to zero in $\mathrm{L}^q(\mathbb{R}^d\times\mathbb{R})$, for some $q>2$. Let $(v_n(\mathbf{x}))$ be an uniformly compactly supported sequence on Ω weakly-$\star$ converging to zero in $\mathrm{L}^\infty(\mathbb{R}^d)$, and let $a:S\to\mathbb{R}^{d\times d}$ be a Borel measurable matrix function such that for a.e. $\lambda\in S$ matrix $a(\lambda)$ is symmetric and positive semi-definite.*

Then for $p=\frac{2q}{q-2}$ there exists a subsequence (not relabeled) and a continuous functional μ on $W^p_\Pi(\Omega,S)$ such that for every $\varphi\in\mathrm{L}^p(\Omega)$ and $\psi\in\mathrm{C}_c(B[0,1]\times S)$ it holds

$$\mu(\varphi\psi) = \lim_{n\to\infty}\int_{\Omega\times S}\varphi(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})}\,d\mathbf{x}\,d\lambda\,. \tag{24}$$

Furthermore, the bound of functional μ is $C_{u,q,2}C_{v,2}$, where $C_{u,q,2}$ is the $\mathrm{L}^q_{\mathbf{x}}(\mathrm{L}^2_\lambda)$-bound of (u_n) and $C_{v,2}$ is the L^2-bound of (v_n).

Proof: First, notice that the mappings

$$\varphi(\mathbf{x})\psi(\boldsymbol{\xi},\lambda) \mapsto \int_{\Omega\times S} \varphi(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})}\, d\mathbf{x}\, d\lambda$$

define a sequence of linear mappings (μ_n) defined on $\widetilde{W}^p_\Pi(\Omega,S)$. We shall prove that the sequence (μ_n) is bounded on $\widetilde{W}^p_\Pi(\Omega,S)$ with respect to the norm $\|\cdot\|_{W^p_\Pi}$. Since $\widetilde{W}^p_\Pi(\Omega,S)$ is dense in $W^p_\Pi(\Omega,S)$ this will imply that (μ_n) is a bounded sequence of linear functionals on $W^p_\Pi(\Omega)$. According to the Banach-Alaoglu-Bourbaki theorem, we conclude that (μ_n) is weakly-$\star$ precompact and a subsequential limit μ will satisfy conditions of the theorem.

Now, notice that any function belonging to $\widetilde{W}^p_\Pi(\Omega,S)$ can be approximated by sums of the form

$$\sum_{j=1}^N \chi_j(\mathbf{x})\psi_j(\boldsymbol{\xi},\lambda)\ ,$$

where $N\in\mathbb{N}$, $\chi_j(\mathbf{x})$, $j=1,\dots,N$, are characteristic measurable functions with disjoint supports, and $\psi_j\in \mathrm{C}^d_c(B[0,1]\times S)$. Thus, it is enough to derive bounds for μ_n on functions of the above form.

By the properties of the commutator given in Corollary 15 we have for a.e. $\lambda\in S$ and any j

$$\lim_{n\to\infty}\int_\Omega \chi_j(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi_j(\pi_P(\cdot,\lambda),\lambda)}\big((1-\chi_j)v_n\big)(\mathbf{x})}\, d\mathbf{x} = 0\ ,$$

where we have used $\chi_j^2=\chi_j$. Thus, as the norm of $\mathcal{A}_{\bar\psi_j(\pi_P(\cdot,\lambda),\lambda)}$ is independent of λ (Lemma 9), by the Lebesgue dominated convergence theorem we get

$$\begin{aligned}&\limsup_{n\to\infty}\Big|\int_{\Omega\times S}\sum_{j=1}^N \chi_j(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi_j(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})}\, d\mathbf{x}\, d\lambda\Big|\\ &\qquad = \limsup_{n\to\infty}\Big|\int_{\Omega\times S}\sum_{j=1}^N \chi_j(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi_j(\pi_P(\cdot,\lambda),\lambda)}(\chi_j\, v_n)(\mathbf{x})}\, d\mathbf{x}\, d\lambda\Big|\ .\end{aligned}$$

Applying the Plancherel formula, the Fubini theorem, and the Cauchy–Bunjakovskij–Schwartz (C–B–S) inequality in λ, the above term is estimated by

$$\limsup_{n\to\infty}\int_{\mathbb{R}^d}\sum_{j=1}^N\int_S\Big|\big(\chi_j\widehat{u_n(\cdot,\lambda)}\big)(\boldsymbol{\xi})\psi_j(\pi_P(\boldsymbol{\xi},\lambda),\lambda)\Big|\,d\lambda\ \big|\widehat{\chi_j v_n}(\boldsymbol{\xi})\big|\,d\boldsymbol{\xi} \tag{25}$$

$$\begin{aligned}&\le \limsup_{n\to\infty}\int_{\mathbb{R}^d}\sum_{j=1}^N\left(\int_S\Big|\big(\chi_j\widehat{u_n(\cdot,\lambda)}\big)(\boldsymbol{\xi})\Big|^2 d\lambda\right)^{1/2}\left(\int_S|\psi_j(\pi_P(\boldsymbol{\xi},\lambda),\lambda)|^2 d\lambda\right)^{1/2}\big|\widehat{\chi_j v_n}(\boldsymbol{\xi})\big|\,d\boldsymbol{\xi}\\ &\le \limsup_{n\to\infty}\int_{\mathbb{R}^d}\sum_{j=1}^N A_j^{1/2}\left(\int_S\Big|\big(\chi_j\widehat{u_n(\cdot,\lambda)}\big)(\boldsymbol{\xi})\Big|^2 d\lambda\right)^{1/2}\big|\widehat{\chi_j v_n}(\boldsymbol{\xi})\big|\,d\boldsymbol{\xi}\,,\end{aligned}$$

where

$$A_j := \sup_{\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}}\int_S|\psi_j(\pi_P(\boldsymbol{\xi},\lambda),\lambda)|^2 d\lambda\,. \tag{26}$$

We continue the estimate by applying first the discrete version of C–B–S inequality, and then its integral version with respect to $\boldsymbol{\xi}$, obtaining that the above term is majorised by

$$\begin{aligned}
&\limsup_{n\to\infty}\int\limits_{\mathbb{R}^d}\Bigg(\sum_{j=1}^N A_j\int\limits_S\Big|\big(\chi_j\widehat{u_n(\cdot,\lambda)}\big)(\boldsymbol{\xi})\Big|^2 d\lambda\Bigg)^{1/2}\Bigg(\sum_{j=1}^N\big|\widehat{\chi_j v_n}(\boldsymbol{\xi})\big|^2\Bigg)^{1/2} d\boldsymbol{\xi}\\
&\qquad\le\limsup_{n\to\infty}\Bigg(\int\limits_{\mathbb{R}^d}\sum_{j=1}^N A_j\int\limits_S\Big|\big(\chi_j\widehat{u_n(\cdot,\lambda)}\big)(\boldsymbol{\xi})\Big|^2 d\lambda d\boldsymbol{\xi}\Bigg)^{1/2}\Bigg(\int\limits_{\mathbb{R}^d}\sum_{j=1}^N\big|\widehat{\chi_j v_n}(\boldsymbol{\xi})\big|^2 d\boldsymbol{\xi}\Bigg)^{1/2}\\
&\qquad=\limsup_{n\to\infty}\Bigg(\int\limits_{\Omega}\sum_{j=1}^N A_j\int\limits_S|(\chi_j u_n)(\mathbf{x},\lambda)|^2 d\lambda d\mathbf{x}\Bigg)^{1/2}\Bigg(\int\limits_{\Omega}\sum_{j=1}^N|(\chi_j v_n)(\mathbf{x})|^2 d\mathbf{x}\Bigg)^{1/2},
\end{aligned}$$

where the Plancherel formula is used in the last equality.

As supports of χ_j are disjoint, we have

$$\int\limits_{\Omega}\sum_{j=1}^N|(\chi_j v_n)(\mathbf{x})|^2 d\mathbf{x}\le\int\limits_{\Omega}|v_n(\mathbf{x})|^2 d\mathbf{x}=\|v_n\|^2_{\mathrm{L}^2(\Omega)},$$

while on the first factor we apply the Hölder inequality $(1/q+1/p=1/2)$ in $\mathbf{x}$:

$$\begin{aligned}
&\Bigg(\int\limits_{\Omega}\sum_{j=1}^N A_j\int\limits_S|(\chi_j u_n)(\mathbf{x},\lambda)|^2 d\lambda d\mathbf{x}\Bigg)^{1/2}\\
&\quad=\Bigg(\int\limits_{\Omega}\|u_n(\mathbf{x},\cdot)\|^2_{\mathrm{L}^2(S)}\Bigg(\sum_{j=1}^N A_j\chi_j(\mathbf{x})\Bigg)d\mathbf{x}\Bigg)^{1/2}\\
&\quad\le\|u_n\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(S))}\Bigg(\int\limits_{\Omega}\Bigg(\sum_{j=1}^N\chi_j(\mathbf{x})\Bigg(\sup_{\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}}\int\limits_S|\psi_j(\pi_P(\boldsymbol{\xi},\lambda),\lambda)|^2 d\lambda\Bigg)\Bigg)^{\frac{p}{2}}d\mathbf{x}\Bigg)^{\frac{1}{p}}\\
&\quad\le\|u_n\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(S))}\Bigg(\int\limits_{\Omega}\Bigg(\sup_{\boldsymbol{\xi}\in\mathbb{R}^d\setminus\{0\}}\Bigg(\int\limits_S\Big|\sum_{j=1}^N\chi_j(\mathbf{x})\psi_j(\pi_P(\boldsymbol{\xi},\lambda),\lambda)\Big|^2 d\lambda\Bigg)^{1/2}\Bigg)^p d\mathbf{x}\Bigg)^{1/p},
\end{aligned}$$

where in the last inequality we have used once more that χ_j have disjoint supports.

Therefore, the final estimate obtained in the above calculations reads

$$\limsup_{n\to\infty}\Big|\int\limits_{\Omega\times S}\sum_{j=1}^N\chi_j(\mathbf{x})u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi_j(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})}\,d\mathbf{x}\,d\lambda\Big|\le C_{u,q,2}\,C_{v,2}\Big\|\sum_{j=1}^N\chi_j\psi_j\Big\|_{W^p_\Pi(\Omega,S)},$$

where $C_{u,q,2}=\limsup_n\|u_n\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(S))}$ and $C_{v,2}=\limsup_n\|v_n\|_{\mathrm{L}^2(\Omega)}$, implying the boundedness of the sequence (μ_n) in $(\widetilde{W^p_\Pi}(\Omega,S),\|\cdot\|_{W^p_\Pi})$. Thus, the sequence is bounded in $W^p_\Pi(\Omega,S)$ as well and, since $W^p_\Pi(\Omega,S)$ is a separable Banach space, the Banach-Alaoglu-Bourbaki theorem provides the statement of the theorem. ❏

Remark 17. In the previous proof, it was only needed that the sequence (u_n) is bounded in $\mathrm{L}^2(\mathbb{R}^{d+1})\cap\mathrm{L}^q(\mathbb{R}^d;\mathrm{L}^2(\mathbb{R}))$ for some $q>2$, so the assumption of the previous theorem could be weakened accordingly.

Remark 18. Let us note that in (25) one could consider integration with respect to $\boldsymbol{\xi}$ only over $|\boldsymbol{\xi}|>M$ for any fixed $M>0$. Indeed, $\widehat{\chi_j v_n}(\boldsymbol{\xi})\to 0$ as $n\to\infty$ for every fixed $\boldsymbol{\xi}\in\mathbb{R}^d$ and

$j \in \{1, 2, \dots, N\}$ since (v_n) is uniformly compactly supported and converges weakly-$\star$ to 0 in $\mathrm{L}^\infty(\mathbb{R}^d)$. On the other hand,

$$\int_S \Big| \big(\chi_j \widehat{u_n(\cdot, \lambda)}\big)(\boldsymbol{\xi}) \psi_j(\pi_P(\boldsymbol{\xi}, \lambda), \lambda) \Big| \, d\lambda \le \|\psi_j\|_{\mathrm{L}^\infty(B[0,1]\times S)} \|u_n\|_{\mathrm{L}^1(\Omega \times S)} < \infty \, .$$

Thus, we can apply the Lebesgue dominated convergence theorem to conclude that the part of (25) in which the integration is over $|\boldsymbol{\xi}| \le M$ converges to zero as $n \to \infty$.

As a consequence this has that in (26) the supremum could be taken only for $|\boldsymbol{\xi}| > M$, implying that μ from Theorem 16 satisfies a sharper estimate:

$$|\langle \mu, \Psi \rangle| \le \left(\int_\Omega \left[\sup_{|\boldsymbol{\xi}|>M} \left(\int_S |\Psi\big(\mathbf{x}, \pi_P(\boldsymbol{\xi}, \lambda), \lambda\big)|^2 \, d\lambda \right)^{1/2} \right]^p d\mathbf{x} \right)^{1/p} , \quad \psi \in W^p_\Pi(\Omega, S) \, , \tag{27}$$

for any fixed $M > 0$.

We are actually able to show the following representation for the functional from the previous theorem for less regular functions with respect to $\mathbf{x}$ and λ.

Corollary 19. *Under the conditions of the previous theorem, let us consider a subsequence (not relabeled) that defines* $\mu \in \Big(W_\Pi^{\frac{2q}{q-2}}(\Omega, S)\Big)'$ *by* (24)*. Then for any* $\varphi \in \mathrm{L}^r(\Omega \times S)$*,* $r > \frac{q}{q-1}$*, and* $\psi \in \mathrm{C}^d_c(B[0,1] \times S)$ *it holds*

$$\mu(\varphi\psi) = \lim_{n\to\infty} \int_{\Omega \times S} \varphi(\mathbf{x}, \lambda) u_n(\mathbf{x}, \lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, d\mathbf{x} d\lambda \, . \tag{28}$$

Proof: In order to prove (28), we shall use a fairly direct approximation argument (see also [42, Theorem 2.2]). To this end, we take a function $\varphi \in \mathrm{L}^r(\Omega \times S)$ and choose its approximation in $\mathrm{L}^{\frac{2q}{q-2}}(\Omega) \times \mathrm{C}^d(S)$ of the form

$$\varphi_s(\mathbf{x}, \lambda) = \sum_{k=1}^s \phi_k(\mathbf{x}) \chi_k(\lambda) \, , \quad \phi_k \in \mathrm{L}^{\frac{2q}{q-2}}(\Omega), \ \chi_k \in \mathrm{C}^d_c(S) \, ,$$

i.e. $\lim_{s\to\infty} \|\varphi - \varphi_s\|_{\mathrm{L}^r(\Omega\times S)} = 0$. Then, we define an extension of μ by

$$\mu(\varphi\psi) := \lim_{s\to\infty} \mu(\varphi_s \psi) \, .$$

By the commutation identity (Corollary 15) it follows

$$\begin{aligned}
\mu(\varphi_s\psi) &= \lim_{n\to\infty} \sum_{k=1}^s \int_{\Omega\times S} \phi_k(\mathbf{x}) u_n(\mathbf{x}, \lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)\chi_k(\lambda)}(v_n)(\mathbf{x})} \, d\mathbf{x} d\lambda \\
&= \lim_{n\to\infty} \sum_{k=1}^s \int_{\Omega\times S} \phi_k(\mathbf{x}) \chi_k(\lambda) u_n(\mathbf{x}, \lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, d\mathbf{x} d\lambda \\
&= \lim_{n\to\infty} \int_{\Omega\times S} \varphi_s(\mathbf{x}, \lambda) u_n(\mathbf{x}, \lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, d\mathbf{x} d\lambda \, ,
\end{aligned}$$

thus, the above definition is equivalent to

$$\mu(\varphi\psi) = \lim_{s\to\infty} \lim_{n\to\infty} \int_{\Omega\times S} \varphi_s(\mathbf{x}, \lambda) u_n(\mathbf{x}, \lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, d\mathbf{x} d\lambda \, . \tag{29}$$

This limit is well-defined as one can see from the Cauchy criterion. Indeed, for $s_1, s_2 \in \mathbb{N}$ by means of the Hölder inequality and the multiplier lemma (Lemma 9) we have

$$\big|\mu(\varphi_{s_2}\psi) - \mu(\varphi_{s_1}\psi)\big| \le C \|\varphi_{s_2} - \varphi_{s_1}\|_{\mathrm{L}^r(\Omega\times S)} \, ,$$

and the constant C is equal to $C_{\bar r,\psi}\,\mathrm{meas}(S)^{\frac{1}{\bar r}}\limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^q(\Omega\times S)}\|v_n\|_{\mathrm{L}^{\bar r}(\Omega)}$, where $\frac{1}{r}+\frac{1}{q}+\frac{1}{\bar r}=1$, and $C_{\bar r,\psi}$ is the $\mathrm{L}^{\bar r}$-bound of the Fourier multiplier operator $\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}$. Since (φ_s) is a Cauchy sequence, the above difference can be made arbitrarily small for s_1, s_2 large enough, hence (29) is well defined. The same analysis leads to

$$\lim_{s\to\infty}\int_{\Omega\times S}\Big(\varphi(\mathbf{x},\lambda)-\varphi_s(\mathbf{x},\lambda)\Big)u_n(\mathbf{x},\lambda)\overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})}\,d\mathbf{x}d\lambda=0\,,$$

and the convergence is uniform with respect to n. Therefore, we can exchange the limits in (29), which proves (28). ❑

Remark 20. The representation (28) holds even for $\tilde\psi(\boldsymbol{\xi}):=2\pi(1-|\boldsymbol{\xi}|)$, which is merely continuous (at the origin $\tilde\psi$ is not smooth).

Indeed, in the construction of the previous corollary we only needed that for a.e. λ and any $p\in(1,\infty)$ mapping $\boldsymbol{\xi}\mapsto\bar\psi\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)$ is an $\mathrm{L}^p(\mathbb{R}^d)$-multiplier, with the norm independent of λ (since this ensures that the statement of Corollary 15 holds as well). By Lemma 10 function $\tilde\psi$ satisfies this requirement.

Let us now introduce a *localisation principle* for functionals μ given by Theorem 16 (see also Corollary 19), which can serve as a way of proving that $\mu\equiv 0$. A similar result holds for arbitrary continuous functionals on $W^p_\Pi(\Omega,S)$ as well.

Lemma 21. *Let us assume that the conditions of Theorem 16 are fulfilled. If function* $F\in\mathrm{C}_0(\mathbb{R}^{d+1}\times B[0,1])$ *is such that for any compact* $K\subseteq S$ *it holds*

$$\lim_{\varepsilon\to 0^+}g(\varepsilon)=0\,, \tag{30}$$

where

$$g(\varepsilon):=\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{|\boldsymbol{\xi}|>1}\mathrm{meas}\Big\{\lambda\in K:\ \Big|F\Big(\mathbf{x},\lambda,\frac{\boldsymbol{\xi}}{\pi_P(\boldsymbol{\xi},\lambda)}\Big)\Big|<\varepsilon\Big\}\,,$$

and

$$F\mu\equiv 0\,, \tag{31}$$

then

$$\mu\equiv 0\,.$$

Proof: Let us take arbitrary $\varepsilon>0$ and $\phi\in\mathrm{C}_c(\Omega\times B[0,1]\times S)$. Denote by K the projection of the support of ϕ to the last variable λ. Applying (31) to $\phi\frac{\bar F}{|F|^2+\varepsilon}$ (which is admissible test function since it is a bounded compactly supported function on $\Omega\times B[0,1]\times S$), we get

$$0=\Big\langle\mu,\phi\frac{|F|^2}{|F|^2+\varepsilon}\Big\rangle=\langle\mu,\phi\rangle-\Big\langle\mu,\phi\Big(\frac{|F|^2}{|F|^2+\varepsilon}-1\Big)\Big\rangle\,.$$

Thus, it is sufficient to show that the second term on the right hand side goes to 0 as $\varepsilon\to 0$. From (27) (applied for $M=1$) we have

$$\begin{aligned}&\Big|\Big\langle\mu,\phi\Big(\frac{|F|^2}{|F|^2+\varepsilon}-1\Big)\Big\rangle\Big|\\&\quad\leq\|\phi\|_{\mathrm{L}^\infty}\left(\int_\Omega\left[\sup_{|\boldsymbol{\xi}|>1}\left(\int_K\Big|\frac{\varepsilon}{|F|^2+\varepsilon}\Big|^2d\lambda\right)^{1/2}\right]^p d\mathbf{x}\right)^{1/p}.\end{aligned}$$

Thus, it is left to prove

$$\lim_{\varepsilon\to 0^+}\left(\int_\Omega\left[\sup_{|\boldsymbol{\xi}|>1}\left(\int_K\Big|\frac{\varepsilon}{|F|^2+\varepsilon}\Big|^2d\lambda\right)^{1/2}\right]^p d\mathbf{x}\right)^{1/p}=0\,. \tag{32}$$

To this end, denote

$$K^\theta(\boldsymbol{\xi},\mathbf{x}) := \Big\{\lambda\in K:\ \Big|F\Big(\mathbf{x},\lambda,\frac{\boldsymbol{\xi}}{\pi_P(\boldsymbol{\xi},\lambda)}\Big)\Big| < \theta\Big\}\,.$$

Let us separately analyse (32) on $K^{\sqrt{\varepsilon}} = K^{\sqrt{\varepsilon}}(\boldsymbol{\xi},\mathbf{x})$ and its complement.

By the assumption (30) we have

$$\begin{aligned}&\int_\Omega\Big[\sup_{|\boldsymbol{\xi}|>1}\Big(\int_{K^{\sqrt{\varepsilon}}}\Big|\frac{\varepsilon}{|F|^2+\varepsilon}\Big|^2\,d\lambda\Big)^{1/2}\Big]^p d\mathbf{x}\\ &\qquad\le \operatorname{meas}(\Omega)\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{|\boldsymbol{\xi}|>1}\sqrt{\operatorname{meas}\big(K^{\sqrt{\varepsilon}}(\boldsymbol{\xi},\mathbf{x})\big)} = \operatorname{meas}(\Omega)\sqrt{g(\sqrt{\varepsilon})}\longrightarrow 0\end{aligned}$$

as $\varepsilon\to 0$.

On the other hand, on the complement we get

$$\begin{aligned}&\int_\Omega\Big[\sup_{|\boldsymbol{\xi}|>1}\Big(\int_{(K\setminus K^{\sqrt{\varepsilon}})}\Big|\frac{\varepsilon}{|F|^2+\varepsilon}\Big|^2\,d\lambda\Big)^{1/2}\Big]^p d\mathbf{x}\\ &\qquad\le \operatorname{meas}(\Omega)\Big\|\frac{\varepsilon}{\sqrt{\varepsilon}+\varepsilon}\Big\|_{\mathrm{L}^2(K\setminus K^{\sqrt{\varepsilon}})}\le \operatorname{meas}(\Omega)\sqrt{\varepsilon\operatorname{meas}(K)}\,.\end{aligned}$$

Therefore, (32) holds, which, by previous observations, provides $\langle\mu,\phi\rangle = 0$, finishing the proof. ❑

In the application of the previous lemma we shall have a specific form of the function F for which the *non-degeneracy assumption* (30) simplifies.

Lemma 22. *Let us assume that the conditions of Theorem 16 are fulfilled, and let us define*

$$F(\mathbf{x},\lambda,\boldsymbol{\xi}) := i\langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle + 2\pi(1-|\boldsymbol{\xi}|)\,,$$

where $f\in\mathrm{L}^r(\Omega\times S;\mathbb{R}^d)$, $r>\frac{q}{q-1}$.

For the function F *condition* (30) *is equivalent to*

$$(\forall K\subset\subset S)\qquad \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{\boldsymbol{\xi}\in\mathrm{S}^{d-1}}\operatorname{meas}\Big\{\lambda\in K:\ \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\Big\} = 0\,. \tag{33}$$

Proof: Suppose that (33) does not hold, i.e. there exists a compact set $K\subseteq S$ such that

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{\boldsymbol{\xi}\in\mathrm{S}^{d-1}}\operatorname{meas}\Big\{\lambda\in K:\ \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\Big\} > 0\,.$$

Since for any $\varepsilon>0$ it holds

$$\Big\{\lambda\in K:\ \Big|F\Big(\mathbf{x},\lambda,\frac{\boldsymbol{\xi}}{\pi_P(\boldsymbol{\xi},\lambda)}\Big)\Big|<\varepsilon\Big\}\supseteq\Big\{\lambda\in K:\ \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\Big\}\,,$$

we get

$$\begin{aligned}g(\varepsilon)&\ge\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{|\boldsymbol{\xi}|>1}\operatorname{meas}\Big\{\lambda\in K:\ \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\Big\}\\ &=\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\sup_{\boldsymbol{\xi}\in\mathrm{S}^{d-1}}\operatorname{meas}\Big\{\lambda\in K:\ \langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle = 0\Big\}>0\,,\end{aligned}$$

implying that (30) does not hold as well.

To prove the opposite, assume that (33) holds but $\lim\limits_{\varepsilon\to 0^+} g(\varepsilon)=0$ fails to hold. This means that there exists $c>0$ such that for some non zero measure set $\Omega\subset\mathbb{R}^d$ for every $\mathbf{x}\in\Omega$ there exists $\boldsymbol{\xi}_n = \boldsymbol{\xi}_n(\mathbf{x}) \in \mathbb{R}^d\backslash\{|\boldsymbol{\xi}| < 1\}$, such that $\operatorname{meas}(K_n) > c$, where

$$K_n := \left\{ \lambda \in K : |\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}_n\rangle| + 2\pi\langle a(\lambda)\boldsymbol{\xi}_n \,|\, \boldsymbol{\xi}_n\rangle < \frac{1}{n}\Big(|\boldsymbol{\xi}_n| + \langle a(\lambda)\boldsymbol{\xi}_n \,|\, \boldsymbol{\xi}_n\rangle\Big) \right\}.$$

The condition defining the set K_n is equivalent to

$$|\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}_n\rangle| + \Big(2\pi - \frac{1}{n}\Big)\langle a(\lambda)\boldsymbol{\xi}_n \,|\, \boldsymbol{\xi}_n\rangle < \frac{1}{n}|\boldsymbol{\xi}_n|\,, \tag{34}$$

i.e. $\lambda \in K$ is contained in K_n if and only if (34) holds. Since $2\pi - \frac{1}{n} > 1$, $n \in \mathbb{N}$, from above we get that for any $\lambda \in K_n$ we have

$$|\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}_n\rangle| + \langle a(\lambda)\boldsymbol{\xi}_n \,|\, \boldsymbol{\xi}_n\rangle < \frac{1}{n}|\boldsymbol{\xi}_n|\,.$$

Dividing by $|\boldsymbol{\xi}_n|$, we obtain

$$\left|\Big\langle f(\mathbf{x},\lambda) \,\Big|\, \frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|}\Big\rangle\right| + |\boldsymbol{\xi}_n|\Big\langle a(\lambda)\frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|} \,\Big|\, \frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|}\Big\rangle < \frac{1}{n}\,. \tag{35}$$

Fix now a non zero measure compact subset $\tilde\Omega \subset \Omega$. Since the sequence $\boldsymbol{\eta}_n(\mathbf{x}) = \boldsymbol{\eta}_n := \frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|} : \tilde\Omega \to \mathrm{S}^{d-1}$ is uniformly continuous, there exits a subsequence (not relabelled) such that $\boldsymbol{\eta}_n \to \boldsymbol{\eta} \in \mathrm{S}^{d-1}$ uniformly on $\tilde\Omega$.

Let us define

$$\varepsilon_n := \operatorname*{ess\,sup}_{\mathbf{x}\in\tilde\Omega}\Big(1 + \max_{\lambda\in K}|f(\mathbf{x},\lambda)| + 2\max_{\lambda\in K}|a(\lambda)|\Big)|\boldsymbol{\eta} - \boldsymbol{\eta}_n|\,,$$

being a sequence of non-negative real numbers converging to zero.

Keeping in mind $|\boldsymbol{\xi}_n| > 1$, from (35) now we get

$$|\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\eta}\rangle| + \langle a(\lambda)\boldsymbol{\eta} \,|\, \boldsymbol{\eta}\rangle < \frac{1}{n} + \varepsilon_n\,,$$

thus

$$K_n \subseteq \tilde K_n := \Big\{\lambda \in K : |\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\eta}\rangle| + \langle a(\lambda)\boldsymbol{\eta} \,|\, \boldsymbol{\eta}\rangle < \frac{1}{n} + \varepsilon_n\Big\}\,,$$

implying

$$\infty > \operatorname{meas}(K) \geq \operatorname{meas}(\tilde K_n) > c > 0 \quad , \ n \in \mathbb{N}\,.$$

Furthermore, $(\tilde K_n)$ is a decreasing sequence of sets, hence

$$\begin{aligned}\operatorname{meas}&\Big\{\lambda \in K :\ \langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\eta}\rangle = \langle a(\lambda)\boldsymbol{\eta} \,|\, \boldsymbol{\eta}\rangle = 0\Big\}\\ &= \operatorname{meas}\Big\{\lambda \in K :\ |\langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\eta}\rangle| + \langle a(\lambda)\boldsymbol{\eta} \,|\, \boldsymbol{\eta}\rangle = 0\Big\} = \lim_{n\to\infty}\operatorname{meas}(K_{2,n}) > c > 0\,,\end{aligned}$$

contradicting (33). ❑

4. Proof of the main theorem

In this section, we shall apply previously developed tools to prove a velocity averaging result for the sequence of equations given in the Introduction:

$$\begin{aligned}\operatorname{div}_{\mathbf{x}}\big(f(\mathbf{x},\lambda)u_n(\mathbf{x},\lambda)\big) &= \operatorname{div}_{\mathbf{x}}\big(\operatorname{div}_{\mathbf{x}}\,(a(\lambda)u_n(\mathbf{x},\lambda))\big)\\ &\qquad + \partial_\lambda G_n(\mathbf{x},\lambda) + \operatorname{div}_{\mathbf{x}}P_n(\mathbf{x},\lambda) \qquad \text{in } \mathcal{D}'(\mathbb{R}^{d+1})\,,\end{aligned} \tag{1}$$

where we assume that conditions (a)–(e) are fulfilled.

Proof of Theorem 1: Let $\Omega \times S \subseteq \mathbb{R}^{d+1}$ be a bounded open subset such that supports of all functions u_n are contained in it, and let us take a bounded sequence of functions (v_n) uniformly compactly supported on Ω and weakly-$\star$ converging to zero in $\mathrm{L}^\infty(\Omega)$, which we take at this moment to be arbitrary, while at the end of the proof the precise choice will be made. Let us pass to a subsequence of both (u_n) and (v_n) (not relabeled) which defines a bounded linear functional $\mu \in \big(W_{\Pi}^{\frac{2q}{q-2}}(\Omega, S)\big)'$ according to Theorem 16, and consider its extension given in Corollary 19.

For arbitrary $\varphi \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}_c^{d+1}(B[0,1] \times S)$ we set

$$\theta_n(\mathbf{x}, \lambda) := \overline{\mathcal{A}_{\frac{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}{|\cdot| + \langle a(\lambda)\cdot\,|\,\cdot\rangle}}(\varphi v_n)(\mathbf{x})}\,. \tag{36}$$

Testing (1) by θ_n, i.e. multiplying by θ_n, integrating over $\Omega \times S$, and applying the integration by parts we get the following:

$$0 = 2\pi \sum_{j=1}^{d} \int_{\Omega\times S} f_j(\mathbf{x},\lambda) u_n(\mathbf{x},\lambda) \overline{\mathcal{A}_{\frac{i\xi_j\bar\psi(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}(\varphi v_n)(\mathbf{x})}\, d\mathbf{x} d\lambda \tag{37}$$

$$- 2\pi \int_{\Omega\times S} u_n(\mathbf{x},\lambda) \overline{\mathcal{A}_{\frac{2\pi\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle\bar\psi(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}(\varphi v_n)(\mathbf{x})}\, d\mathbf{x} d\lambda \tag{38}$$

$$- \int_{\Omega\times S} G_n(\mathbf{x},\lambda) \overline{\mathcal{A}_{\partial_\lambda \frac{\bar\psi(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}(\varphi v_n)(\mathbf{x})}\, d\mathbf{x} d\lambda \tag{39}$$

$$- \sum_{j=1}^{d} \int_{\Omega\times S} P_j^n(\mathbf{x},\lambda) \overline{\mathcal{A}_{\frac{2\pi i\xi_j\bar\psi(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}(\varphi v_n)(\mathbf{x})}\, d\mathbf{x} d\lambda\,, \tag{40}$$

where we have used $\partial_{x_j}\mathcal{A}_\psi = \mathcal{A}_{2\pi i\xi_j\psi}$, according to our definition of the Fourier transform. Line (39) is to be understood as

$$\int_S \Big\langle G_n(\cdot,\lambda), \overline{\mathcal{A}_{\partial_\lambda \frac{\bar\psi(\pi_P(\cdot,\lambda),\lambda)}{|\cdot| + \langle a(\lambda)\cdot\,|\,\cdot\rangle}}(\varphi v_n)} \Big\rangle\, d\lambda\,,$$

where $\langle\cdot,\cdot\rangle$ represents the dual product between $\mathrm{W}_{loc}^{-\frac{1}{2},r}(\mathbb{R}^d)$ and $\mathrm{W}_c^{\frac{1}{2},r'}(\mathbb{R}^d)$.

Let us consider term by term in the above expression as n goes to infinity along the chosen subsequence.

Symbols of the Fourier multipliers in (37) and (38) can be rewritten as

$$\frac{i\xi_j\bar\psi\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} = \big(\overline{\psi_j\psi}\big)\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)$$
$$\frac{2\pi\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle\bar\psi\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} = \big(\overline{\tilde\psi\psi}\big)\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)\,,$$

where $\psi_j(\boldsymbol{\xi}) := -i\xi_j$ and $\tilde\psi(\boldsymbol{\xi}) := 2\pi(1 - |\boldsymbol{\xi}|)$. Thus, by applying first Corollary 15 in order to move φ outside of the Fourier multiplier operators, and then Corollary 19 (see also Remark 20), the limit of the sum of (37) and (38) is equal to

$$-2\pi\Big\langle \mu, F(\mathbf{x},\boldsymbol{\xi},\lambda)\varphi(\mathbf{x})\psi(\boldsymbol{\xi},\lambda)\Big\rangle\,, \tag{41}$$

where $F(\mathbf{x},\boldsymbol{\xi},\lambda) = i\langle f(\mathbf{x},\lambda)\,|\,\boldsymbol{\xi}\rangle + 2\pi\big(1 - |\boldsymbol{\xi}|\big)$.

Unlike the situation with (37) and (38), term (40) is zero at the limit $n \to \infty$. Indeed, P_j^n strongly converges in $\mathrm{L}_{loc}^{p_0}(\mathbb{R}^d \times \mathbb{R})$, while $\mathcal{A}_{(\overline{\psi_j\psi})(\pi_P(\cdot,\cdot),\cdot)}(\varphi v_n)$ weakly converges to zero in $\mathrm{L}^{p_0'}(\Omega \times S)$ by Lemma 9, and the integration is over relatively compact set $\Omega \times S$.

The symbol appearing in (39) we divide into two parts, namely

$$\frac{(\partial_\lambda\bar\psi)\,(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle} \tag{42}$$

and

$$\partial_\lambda\left(\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\right)\left(\bar\psi\big(\pi_P(\boldsymbol{\xi},\lambda),\lambda\big)+\sum_{j=1}^d\big(\xi_j\partial_{\xi_j}\psi(\boldsymbol{\xi},\lambda)\big)\circ\big(\pi_p(\boldsymbol{\xi},\lambda)\big)\right). \tag{43}$$

Let us study first the part of (39) associated to (42).

By Lemma 35 (given in the Appendix), Lemma 9 and the Lebesgue dominated convergence theorem (applied for the integration in λ) we have for any $r\in[1,\infty)$

$$\mathcal{A}_{\frac{(\partial_\lambda\bar\psi)(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi};\boldsymbol{\xi}\rangle}}(\varphi v_n)=\mathcal{A}_{\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi};\boldsymbol{\xi}\rangle}}\Big(\mathcal{A}_{(\partial_\lambda\bar\psi)(\pi_P(\boldsymbol{\xi},\lambda),\lambda)}(\varphi v_n)\Big)\rightharpoonup 0\quad\text{weakly in } \mathrm{L}^r(S;\mathrm{W}^{1,r}(\Omega)),$$

where we have used that $\Omega\times S$ is relatively compact. This together with the assumption of (G_n) implies the convergence to zero of the part of (39) associated to (42).

The Fourier multiplier operator associated to the second factor of (43) is bounded on $\mathrm{L}^r(\mathbb{R}^d)$ for any $r\in(1,\infty)$ uniformly in λ (Lemma 9), while by Lemma 11 the first factor defines a bounded operator (uniformly in λ) from $\mathrm{L}^r(\mathbb{R}^d)$ to $\mathrm{W}^{\frac12,r}(\mathbb{R}^d)$, for any $r\in(1,\infty)$. Thus, the overall conclusion is that for any $r\in(1,\infty)$

$$\mathcal{A}_{\partial_\lambda\left(\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}\right)\left(\bar\psi(\pi_P(\boldsymbol{\xi},\lambda),\lambda)+\sum_{j=1}^d\left(\xi_j\partial_{\xi_j}\psi(\boldsymbol{\xi},\lambda)\right)\circ(\pi_p(\boldsymbol{\xi},\lambda))\right)}(\varphi v_n)\rightharpoonup 0\,,$$

weakly in $\mathrm{L}^r(S;\mathrm{W}^{\frac12,r}(\Omega))$, implying the convergence to zero of the part of (39) associated to (43).

Collecting the previous considerations, we get from (37)–(40) after letting $n\to\infty$ that (41) is the only non-trivial term, thus we have:

$$\Big\langle\mu, F(\mathbf{x},\boldsymbol{\xi},\lambda)\varphi(\mathbf{x})\psi(\boldsymbol{\xi},\lambda)\Big\rangle=0\,.$$

Since F satisfies the non-degeneracy assumption (33) (see Lemma 22), by Lemma 21 we conclude from above that

$$\mu\equiv 0\,.$$

Let us assume that u_n is real valued (if not, we just apply this procedure to the real and imaginary parts of u_n separately). Let us take arbitrary real valued functions $\varphi\in\mathrm{C}_c(\Omega)$ and $\rho\in\mathrm{C}_c(S)$. As $\big(\mathrm{sgn}(\int_S\rho(\lambda)u_n(\mathbf{x},\lambda)\,d\lambda)\big)$ is bounded in $\mathrm{L}^\infty(\Omega)$, it has a weakly-$\star$ converging subsequence, whose limit we denote by $V\in\mathrm{L}^\infty(\Omega)$. We pass to that subsequence (not relabeled), and choose for v_n in (36):

$$v_n(\mathbf{x})=\varphi(\mathbf{x})\left(\mathrm{sgn}\left(\int_S\rho(\lambda)u_n(\mathbf{x},\lambda)\,d\lambda\right)-V(\mathbf{x})\right)\,.$$

As the subsequence defines the same functional μ, by Theorem 16 we conclude

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\varphi(\mathbf{x})^2\left|\int_S\rho(\lambda)u_n(\mathbf{x},\lambda)d\lambda\right|\,d\mathbf{x}\\
&\quad=\lim_{n\to\infty}\int_\Omega\varphi(\mathbf{x})^2\left(\int_S\rho(\lambda)u_n(\mathbf{x},\lambda)d\lambda\right)\mathrm{sgn}\left(\int_S\rho(\lambda)u_n(\mathbf{x},\lambda)d\lambda\right)d\mathbf{x}\\
&\quad=\lim_{n\to\infty}\int_{\Omega\times S}\varphi(\mathbf{x})\rho(\lambda)u_n(\mathbf{x},\lambda)v_n(\mathbf{x})\,d\mathbf{x}\,d\lambda\\
&\quad=\langle\mu,\rho\varphi\otimes 1\rangle=0\,,
\end{aligned}$$

where in the second equality we have used that (u_n) converges weakly to zero. Thus, the proof is over. ❑

We note that matrix function (8) that we have provided in Example 4(b) is irregular only in one point. These situations we can handle by cutting off isolated singularities using appropriate λ-compactly supported functions. Let us briefly explain how to prove Corollary 2.

Proof of Corollary 2: Take a dense countable set of $\lambda_0 \in S$ which satisfy ($\tilde{\rm b}$) denoted by $\tilde{S}$. Fix $\lambda_0 \in \tilde{S}$. Then, in the proof of Theorem 1 given above, we simply take $\rho \in {\rm C}^1_c(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, where $\varepsilon = \varepsilon(\lambda_0)$, to obtain (6). Since (u_n) is bounded in ${\rm L}^1(\Omega \times S)$ (we have boundedness even in ${\rm L}^q(\Omega \times S)$, $q > 2$), then for any $\rho \in {\rm C}^1_c(S)$ we can take $\rho = \rho(\lambda)\chi_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}(\lambda)$, where $\chi_{(\lambda_0-\varepsilon,\lambda_0+\varepsilon)}$ is the characteristic function of the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.

Since $\tilde{S}$ is countable, we can take a sequence in (6) independent of $\lambda_0 \in \tilde{S}$. Furthermore, since $\tilde{S}$ is dense in S, then $\cup_{\lambda_0 \in \tilde{S}}[\lambda_0 - \varepsilon(\lambda_0), \lambda_0 + \varepsilon(\lambda_0)] \supseteq S \backslash E$, where E is the set of measure zero out of which ($\tilde{\rm b}$) holds. From here, the statement of the corollary immediately follows (since for every $\rho \in {\rm C}^1_c(S)$ and every $\lambda_0 \in \tilde{S}$, the sequence $(\int_{\mathbb{R}} \rho(\lambda)\chi_{[\lambda_0-\varepsilon,\lambda_0+\varepsilon]}(\lambda)u_n(\cdot,\lambda)d\lambda)$ converges along the fixed subsequence and segments $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ cover entire S except for the zero measure set). ❑

Remark 23. With the exception of Lemma 11, all other (multiplier) results used in the above proof of Theorem 1 holds under a weaker assumption on a:

b') $a \in {\rm C}^{0,1}(S; \mathbb{R}^{d\times d})$ is such that, for every $\lambda \in S$, $a(\lambda)$ is a symmetric and positive semi-definite matrix.

Thus, a sufficient assumption on the diffusion matrix a under which the statement of Theorem 1 holds is that a satisfies (b') and that (39) tends to 0, as $n \to \infty$. Moreover, it is enough to have local estimates of the multipliers with respect to λ.

To be more specific, a possible weakening of assumption (b) which preserves the statement of Theorem 1 is:

b") a satisfies (b') and for a.e. $\lambda_0 \in S$ there exists its neighborhood $U(\lambda_0)$ such that for any $\lambda \in U(\lambda_0)$ and any $p \in (1, \infty)$

$$\boldsymbol{\xi} \mapsto \frac{|\boldsymbol{\xi}|^{1/2}\langle a'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle}{\big(|\boldsymbol{\xi}| + \langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle\big)^2}$$

is an ${\rm L}^p$-multiplier, with the norm uniformly bounded with respect to $\lambda \in U(\lambda_0)$.

By lemmata 9 and 11 it is clear, which was also used in the previous proof, that (b") is implied by ($\tilde{\rm b}$) (see Corollary 2). However, (b') does not imply (b") in general. For instance, just consider (9) from the Introduction.

Although at this moment we cannot obtain the result by imposing only (b') on the diffusion matrix a, while keeping all the other assumptions intact (see the previous remark), under stronger assumptions on (G_n), given in Corollary 3, that can be done. A proof of Corollary 3 we briefly explain below.

Proof of Corollary 3: If we replace (d) by

d') $G_n \to 0$ strongly in ${\rm L}^{r_0}_{loc}(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}_\lambda)$ for some $r_0 \in (1, \infty)$;

then it is sufficient that (43) is an ${\rm L}^p$-multiplier, $p \in (1, \infty)$, which is ensured by Lemma 9. Thus, the statement of Corollary 3 holds. ❑

If (u_n) has a compact support only with respect to λ, the statement of the previous theorem still holds. Indeed, one just need to test (1) by $\tilde{\varphi}\theta_n$ instead of θ_n (given by (36)) for an arbitrary $\tilde{\varphi} \in {\rm C}_c(\mathbb{R}^d)$. By repeating the rest of the analysis of the proof of Theorem 1 we obtain that the functional μ from Theorem 16 corresponding to $(\tilde{\varphi}u_n)$ equals zero, implying the strong convergence to zero in ${\rm L}^1_{loc}(\mathbb{R}^d)$ of $(\int_{\mathbb{R}} \rho(\lambda)\tilde{\varphi}(\mathbf{x})u_n(\mathbf{x},\lambda)d\lambda)$ for any $\rho \in {\rm C}_c(\mathbb{R})$. Due to arbitrariness of $\tilde{\varphi}$ we get the claim which we formulate in the following corollary.

Corollary 24. *Let $d \geq 2$ and let $u_n \rightharpoonup 0$ in ${\rm L}^q_{loc}(\mathbb{R}^d \times \mathbb{R})$, for some $q > 2$, is uniformly compactly supported with respect to λ on $S \subset\subset \mathbb{R}$. Let (u_n) satisfies the sequence of equations* (1) *whose coefficients satisfy conditions (b), (d), (e), and* c') *There exists* $p > \frac{q}{q-1}$ *(*$p > 1$ *if* $q = \infty$*) such that for any compacts* $K_1 \subseteq \mathbb{R}^d$ *and* $K_2 \subseteq S$ *it holds* $f \in \mathrm{L}^p(K_1 \times S; \mathbb{R}^d)$ *and*

$$\operatorname*{ess\,sup}_{\mathbf{x}\in K_1} \sup_{\boldsymbol{\xi}\in \mathrm{S}^{d-1}} \operatorname{meas}\big\{\lambda \in K_2 : \langle f(\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\big\} = 0\,.$$

Then there exists a subsequence $(u_{n'})$ *such that for any* $\rho \in \mathrm{C}_c(\mathbb{R})$*,*

$$\int_{\mathbb{R}} \rho(\lambda) u_{n'}(\mathbf{x},\lambda)\, d\lambda \longrightarrow 0 \quad \textit{strongly in } \mathrm{L}^1_{loc}(\mathbb{R}^d)\,.$$

5. Degenerate parabolic equation with rough coefficients – existence proof

In this section, we prove existence of a weak solution to the Cauchy problem for the advection-diffusion equation:

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},u) = D^2 \cdot A(u) \qquad , \ (t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^d =: \mathbb{R}^{d+1}_+ \,, \tag{44}$$

$$u|_{t=0} = u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)\,. \tag{45}$$

where $D^2 \cdot A(u) = \sum_{i,j} \partial^2_{x_i x_j} [A(u)]_{ij}$. The equation describes a flow governed by

- the convection effects (bulk motion of particles) which are represented by the first order terms;
- diffusion effects which are represented by the second order term and the matrix $A(\lambda) = [A_{ij}(\lambda)]_{i,j=1,\dots,d}$ (more precisely its derivative with respect to λ; see (46) below) describes direction and intensity of the diffusion;

The equation is degenerate in the sense that the derivative of the diffusion matrix A' can be equal to zero in some direction. Roughly speaking, if this is the case, i.e. if for some vector $\boldsymbol{\xi} \in \mathbb{R}^d$ we have $\langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0$, then diffusion effects do not exist at the point $\mathbf{x}$ for the state λ in the direction $\boldsymbol{\xi}$.

Recently, several existence results for (44) in the case when the coefficients are irregular were obtained. In [40, 49, 53] the authors considered ultra-parabolic equations, while in [36] a degenerate parabolic equation was considered and a similar result as in Theorem 28 below is obtained.

Roughly speaking, in [36], the authors had the assumptions that the flux $\mathfrak{f}(t,\mathbf{x},\lambda)$ is merely continuous with respect to λ and $\max\limits_{|u|<M} |\mathfrak{f}(\mathbf{x},u)| \in \mathrm{L}^2_{loc}(\mathbb{R}^d)$ for every $M > 0$. However, we still generalise this result by assuming the following for the coefficients of (44) (keep in mind that we need only L^p, $p > 1$ assumptions on the flux unlike the L^2 assumptions from [36]):

i) There exist $\alpha, \beta \in \mathbb{R}$ such that the initial data $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ is bounded between α and β and the flux equals zero at $\lambda = \alpha$ and $\lambda = \beta$:

$$\alpha \le u_0(\mathbf{x}) \le \beta \quad \text{and} \quad \mathfrak{f}(t,\mathbf{x},\alpha) = \mathfrak{f}(t,\mathbf{x},\beta) = 0 \ \text{ a.e. } \ (t,\mathbf{x}) \in \mathbb{R}^{d+1}_+\,.$$

ii) The convective term $\mathfrak{f}(t,\mathbf{x},\lambda)$ belongs to $\mathrm{C}^1([\alpha,\beta]; \mathrm{L}^p_{loc}(\mathbb{R}^{d+1}_+))$ for some $p > 1$, and

$$\operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},\lambda) \in \mathcal{M}(\mathbb{R}^{d+1}_+ \times [\alpha,\beta])\,,$$

where $\mathcal{M}(X)$ denotes the space of Radon measures on $X \subseteq \mathbb{R}^d$.

iii) The matrix $A(\lambda) \in \mathrm{C}^{1,1}([\alpha,\beta]; \mathbb{R}^{d\times d})$ is symmetric and non-decreasing with respect to $\lambda \in [\alpha,\beta]$, i.e. the (diffusion) matrix $a(\lambda) := A'(\lambda)$ satisfies

$$\langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle \ge 0\,,$$

and a satisfies (b) from the Introduction.

iv) $f := \partial_\lambda \mathfrak{f}$ and $a = A'$ satisfy non-degeneracy assumption: for any compact $K \subseteq \mathbb{R}^{d+1}_+$ it holds

$$\operatorname*{ess\,sup}_{(t,\mathbf{x})\in K} \sup_{(\tau,\boldsymbol{\xi})\in S^d} \operatorname{meas}\big\{\lambda \in [\alpha,\beta] : \tau + \langle f(t,\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\big\} = 0 .$$

Remark that equation (44) can be rewritten in the standard (more usual) form as follows (cf. [18]):

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathfrak{f}(t,\mathbf{x},u) = \operatorname{div}_{\mathbf{x}}(a(u)\nabla_{\mathbf{x}} u). \tag{46}$$

Thus, by proving existence of solutions to equation (44), we shall prove existence of solutions to Cauchy problems for a degenerate parabolic equation in the standard form (46).

Let us first recall the notion of entropy-solutions for (44) (see [47] for the hyperbolic conservation law).

Definition 25. A measurable function u defined on $\mathbb{R}^+ \times \mathbb{R}^d$ is called a quasi-solution to (44) if $\mathfrak{f}(t,\mathbf{x},u(t,\mathbf{x})) \in L^1_{loc}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$, $A(u(t,\mathbf{x})) \in L^1_{loc}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d\times d})$, and for a.e. $\lambda \in \mathbb{R}$ the Kružkov-type entropy equality holds

$$\begin{aligned}\partial_t|u-\lambda| + &\operatorname{div}_{\mathbf{x}}\left[\operatorname{sgn}(u-\lambda)(\mathfrak{f}(t,\mathbf{x},u)-\mathfrak{f}(t,\mathbf{x},\lambda))\right] \\ &- D^2\cdot\left[\operatorname{sgn}(u-\lambda)(A(u)-A(\lambda))\right] = -\zeta(t,\mathbf{x},\lambda),\end{aligned} \tag{47}$$

where $\zeta \in C(\mathbb{R}_\lambda; w\star-\mathcal{M}(\mathbb{R}^{d+1}_+))$ is a non-negative functional which we call the quasi-entropy defect measure.

Remark 26. Remark that for a regular flux $\mathfrak{f}$, the measure $\zeta(t,\mathbf{x},\lambda)$ can be rewritten in the form $\zeta(t,\mathbf{x},\lambda) = \bar{\zeta}(t,\mathbf{x},\lambda) + \operatorname{sgn}(u-\lambda)\operatorname{div}_{\mathbf{x}}\mathfrak{f}(t,\mathbf{x},\lambda)$, for a measure $\bar{\zeta}$. If $\bar{\zeta}$ is non-negative, then the quasi-solution u is an entropy solution to (44). For the uniqueness of such entropy solution, we additionally need the chain rule [18, 17].

From the notion of quasi-solution, the following kinetic formulation can be proved (see also [55, (4.4)]).

Theorem 27. *If function u is a quasi-solution to* (44)*, then the function*

$$h(t,\mathbf{x},\lambda) = \operatorname{sgn}(u(t,\mathbf{x})-\lambda) = -\partial_\lambda|u(t,\mathbf{x})-\lambda| \tag{48}$$

is a weak solution to the following linear equation:

$$\partial_t h + \operatorname{div}_{\mathbf{x}}\left(f(t,\mathbf{x},\lambda)h\right) - D^2\cdot[a(\lambda)h] = \partial_\lambda\zeta(t,\mathbf{x},\lambda)\,, \tag{49}$$

where $f = \partial_\lambda\mathfrak{f}$ and $a = A'$.

Proof: It is enough to find derivative of (47) with respect to $\lambda \in \mathbb{R}$ to obtain (49). ❑

The main theorem of the section is the following.

Theorem 28. *Let $d \geq 1$, and let $u_0, \mathfrak{f}$ and A satisfy conditions (i)–(iv) above.*

Then there exists a quasi-solution to (44) *augmented with the initial condition* (45).

Proof: Consider the sequence of admissible solutions to the following regularised Cauchy problems

$$\begin{aligned}\partial_t u_n + \operatorname{div}_{\mathbf{x}}\mathfrak{f}_n(t,\mathbf{x},u_n) =& D^2\cdot A(u_n). \\ u_n|_{t=0} =& u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d),\end{aligned}$$

where $\mathfrak{f}_n$ is a smooth regularisation with respect to $(t,\mathbf{x})$ of $\mathfrak{f}$ such that

$$\mathfrak{f}_n(t,\mathbf{x},\alpha) = \mathfrak{f}_n(t,\mathbf{x},\beta) = 0\,, \tag{50}$$

and for any compact $K \subseteq \mathbb{R}^{d+1}_+$

$$\lim_{n\to\infty}\|\mathfrak{f}_n-\mathfrak{f}\|_{L^p(K\times[\alpha,\beta])} = \lim_{n\to\infty}\|\partial_\lambda\mathfrak{f}_n-\partial_\lambda\mathfrak{f}\|_{L^p(K\times[\alpha,\beta])} = 0\,. \tag{51}$$

Notice that we can simply take for $\mathfrak{f}_n$ the convolution of $\mathfrak{f}$ with a standard mollifier. It is well known that there exists a solution u_n to such equation satisfying conditions (47) with $\mathfrak{f}$ replaced by $\mathfrak{f}_n$ (see [18] where existence was shown under much more restrictive conditions), i.e. the following kinetic formulation holds (see Theorem 27):

$$\partial_t h_n + \operatorname{div}\left(f_n(t,\mathbf{x},\lambda)h_n\right) - D^2 \cdot [a(\lambda)h_n] = \partial_\lambda \zeta_n(t,\mathbf{x},\lambda) \tag{52}$$

where $f_n(t,\mathbf{x},\lambda) = \partial_\lambda \mathfrak{f}_n(t,\mathbf{x},\lambda)$, (ζ_n) is a sequence of non-negative entropy defect measures (see Remark 26), and $h_n(t,\mathbf{x},\lambda) = \operatorname{sgn}(u_n(t,\mathbf{x}) - \lambda)$. According to (i) and (50), we know that (u_n) remains bounded between α and β and therefore, the sequence (ζ_n) is bounded in $\mathrm{C}(\mathbb{R}_\lambda; w - \star\mathcal{M}(\mathbb{R}^{d+1}_+))$. This implies that (ζ_n) is actually strongly precompact in $\mathrm{L}^r_{loc}(\mathbb{R}_\lambda; \mathrm{W}^{-\frac{1}{2},q}_{loc}(\mathbb{R}^{d+1}_+))$ for any $r \geq 1$ and $q \in \left[1, \frac{d+1}{d+1-\frac{1}{2}}\right)$ (one can prove this in the same manner as [25, Theorem 1.6] using that $\mathrm{W}^{\frac{1}{2},s}(\mathbb{R}^{d+1})$ is compactly embedded into $\mathrm{C}(Cl\,K)$, $K \subset\subset \mathbb{R}^{d+1}$, for $\frac{s}{2} > d+1$), and let us denote by ζ the limit.

Let us pass to a subsequence (not relabelled) such that (h_n) converges weakly-$\star$ to h in $\mathrm{L}^\infty(\mathbb{R}^{d+1}_+ \times \mathbb{R})$. Due to linearity of (52), we then have

$$\begin{aligned} &\partial_t w_n + \operatorname{div}\left(f(t,\mathbf{x},\lambda)w_n\right) - D^2 \cdot [a(\lambda)w_n] \\ &\qquad = \operatorname{div}\left((f(t,\mathbf{x},\lambda) - f_n(t,\mathbf{x},\lambda))h_n\right) + \partial_\lambda \gamma_n(t,\mathbf{x},\lambda)\,, \end{aligned} \tag{53}$$

where $w_n = h_n - h$ and $\gamma_n = \zeta_n - \zeta$, and both sequences converge to zero (the first convergence is weak-$\star$ in $\mathrm{L}^\infty(\mathbb{R}^{d+1}_+ \times \mathbb{R})$, and the latter strong in $\mathrm{L}^{r_0}_{loc}(\mathbb{R}_\lambda; \mathrm{W}^{-\frac{1}{2},r_0}_{loc}(\mathbb{R}^{d+1}_+)$ for any $r_0 \in (1, \frac{2d+2}{2d+1})$).

Due to the boundedness of (u_n), (w_n) is clearly uniformly compactly supported with respect to λ on $[\alpha,\beta]$. As we also have (51), (53) clearly satisfies conditions of Corollary 24 with $q = \infty$ (see also Remark 6). Therefore, on a subsequence (not relabeled), $(\int_\mathbb{R} \rho(\lambda) w_n(t,\mathbf{x},\lambda)d\lambda)$ converges to zero in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$ for any $\rho \in \mathrm{C}_c(\mathbb{R})$. Due to density arguments, we can insert $\rho(\lambda) = \chi_{[\alpha,\beta]}$, obtaining

$$\begin{aligned} 2u_n(t,\mathbf{x}) - \alpha - \beta &= \int_\alpha^{u_n(t,\mathbf{x})} d\lambda - \int_{u_n(t,\mathbf{x})}^\beta d\lambda \\ &= \int_\alpha^\beta \operatorname{sgn}(u_n(t,\mathbf{x}) - \lambda)\, d\lambda \overset{n\to\infty}{\longrightarrow} \int_\alpha^\beta h(t,\mathbf{x},\lambda)\, d\lambda\,, \end{aligned}$$

where the latter convergence is in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$. Therefore, (u_n) strongly converges in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$ toward

$$u(t,\mathbf{x}) := \frac{\int_\alpha^\beta h(t,\mathbf{x},\lambda)d\lambda\, + \alpha + \beta}{2}\,.$$

The function u is a quasi-solution to (44)–(45). ❑

6. Existence of traces for quasi-solutions to (44)

In this section, we shall prove existence of strong traces for quasi-solutions of (44). Let us first formally introduce the notion of traces.

Definition 29. Let $u \in \mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$. A locally integrable function u_0 defined on $\mathbb{R}^d$ is called the strong trace of u at $t = 0$ if $\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0$ in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$, i.e. for some set $E \subseteq (0,\infty)$ of full Lebesgue measure and any relatively compact set $K \subset\subset \mathbb{R}^d$ it holds

$$\lim_{E \ni t \to 0} \|u(t,\cdot) - u_0\|_{\mathrm{L}^1(K)} = 0\,. \tag{54}$$

The following theorem holds.

Theorem 30. *Let $\mathfrak{f} \in \mathrm{C}^1(\mathbb{R};\mathbb{R}^d)$ and let $A \in \mathrm{C}^{1,1}(\mathbb{R};\mathbb{R}^{d\times d})$ be such that there exists $\sigma \in \mathrm{C}^{0,1}(\mathbb{R};\mathbb{R}^{d\times d})$ such that for any $\lambda \in \mathbb{R}$ we have $a(\lambda) := A'(\lambda) = \sigma(\lambda)^T\sigma(\lambda)$. Moreover, assume that the non-degeneracy condition is satisfied: for any compact $K \subseteq \mathbb{R}$,*

$$\sup_{\boldsymbol{\xi}\in \mathrm{S}^{d-1}} \operatorname{meas}\Big\{\lambda \in K :\ \langle a(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\Big\} = 0\,, \tag{55}$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$ centered at the origin.

Then any bounded quasi-solution $u \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+)$ to (44) admits the strong trace at $t = 0$, i.e. there exists $u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)$ such that

$$\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0$$

strongly in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$.

The structure of proof can be presented as follows.

- We prove existence of weak traces.
- We introduce the scaling $t = \frac{\hat t}{m}$, $x_1 = y_1 + \frac{\hat x_1}{\sqrt{m}}$, $x_2 = y_2 + \frac{\hat x_2}{\sqrt{m}}, \dots, x_d = y_d + \frac{\hat x_d}{\sqrt{m}}$ where $\mathbf{y} \in \mathbb{R}^d$ is a fixed vector.
- We obtain a degenerate parabolic transport equation and we apply the velocity averaging result.
- From the previous item, we conclude the existence of the strong traces.

in accordance with the described strategy, let us first show existence of the weak traces.

Proposition 31. *Let $h \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+ \times \mathbb{R})$ be a distributional solution to (49) and let us define*

$$\begin{aligned} E = \Big\{t \in \mathbb{R}^+ :\ &(t,\mathbf{x},\lambda) \text{ is a Lebesgue point of} \\ &h(t,\mathbf{x},\lambda) \text{ for a.e. } (\mathbf{x},\lambda) \in \mathbb{R}^d\times\mathbb{R}\Big\}\,. \end{aligned} \tag{56}$$

Then there exists $h_0 \in \mathrm{L}^\infty(\mathbb{R}^{d+1})$, such that

$$h(t,\cdot,\cdot) \rightharpoonup h_0\,,\ \text{weakly-}\star \text{ in } \mathrm{L}^\infty(\mathbb{R}^{d+1})\,,\ \text{as } t\to 0\,,\ t\in E\,.$$

Proof: Note first that E is of full measure. Since $h \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+\times\mathbb{R})$, the family $\{h(t,\cdot,\cdot)\}_{t\in E}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^{d+1})$. Due to the weak-$\star$ precompactness of $\mathrm{L}^\infty(\mathbb{R}^{d+1})$, there exists a sequence $\{t_m\}_{m\in\mathbb{N}}$ in E such that $t_m \to 0$ as $m\to\infty$, and $h_0 \in \mathrm{L}^\infty(\mathbb{R}^{d+1})$, such that

$$h(t_m,\cdot,\cdot) \rightharpoonup h_0\,,\ \text{weakly-}\star \text{ in } \mathrm{L}^\infty(\mathbb{R}^{d+1}),\ \text{as } m\to\infty\,. \tag{57}$$

For $\phi \in C^\infty_c(\mathbb{R}^d)$, $\rho \in \mathrm{C}^1_c(\mathbb{R})$, denote

$$I(t) := \int_{\mathbb{R}^{d+1}} h(t,\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda\,,\quad t\in E\,.$$

With this notation, (57) means that

$$\lim_{m\to\infty} I(t_m) = \int_{\mathbb{R}^{d+1}} h_0(\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda =: I(0)\,. \tag{58}$$

Now, fix $\tau \in E$ and notice that for the regularization $I_\varepsilon = I\star\omega_\varepsilon$, where ω_ε is the standard convolution kernel, it holds

$$\lim_{\varepsilon\to 0} I_\varepsilon(\tau) = I(\tau).$$

Then, fix $m_0 \in \mathbb{N}$, such that $E \ni t_m \leq \tau$, for $m \geq m_0$, and remark that

$$\begin{aligned}
I(\tau) - I(t_m) &= \lim_{\varepsilon\to 0} \int_{t_m}^{\tau} I'_\varepsilon(t)\,dt \\
&= \sum_{j=1}^{d} \int_{(t_m,\tau]\times\mathbb{R}^{d+1}} h(t,\mathbf{x},\lambda) f_j(t,\mathbf{x},\lambda)\rho(\lambda)\partial_{x_j}\phi(\mathbf{x})\,dt\,d\mathbf{x}\,d\lambda \\
&\quad - \sum_{j,k=1}^{d} \int_{(t_m,\tau]\times\mathbb{R}^{d+1}} h(t,\mathbf{x},\lambda) a_{jk}(\lambda)\rho(\lambda)\partial_{x_j x_k}\phi(\mathbf{x})\,dt\,d\mathbf{x}\,d\lambda \\
&\quad - \int_{(t_m,\tau]\times\mathbb{R}^{d+1}} \phi(\mathbf{x})\rho'(\lambda)\,d\gamma(t,\mathbf{x},\lambda)\,,
\end{aligned}$$

where we have used that h is a distributional solution to (49). Hence, passing to the limit as $m \to \infty$, and having in mind (58) and the fact that γ is locally finite up to the boundary $t = 0$, we obtain

$$\begin{aligned}
I(\tau) - I(0) &= \sum_{j=1}^{d} \int_{(0,\tau]\times\mathbb{R}^{d+1}} h(t,\mathbf{x},\lambda) f_j(t,\mathbf{x},\lambda)\rho(\lambda)\partial_{x_j}\phi(\mathbf{x})\,dt\,d\mathbf{x}\,d\lambda \\
&\quad - \sum_{j,k=1}^{d} \int_{(0,\tau]\times\mathbb{R}^{d+1}} h(t,\mathbf{x},\lambda) a_{jk}(\lambda)\rho(\lambda)\partial_{x_j x_k}\phi(\mathbf{x})\,dt\,d\mathbf{x}\,d\lambda \\
&\quad - \int_{(0,\tau]\times\mathbb{R}^{d+1}} \phi(\mathbf{x})\rho'(\lambda)\,d\gamma(t,\mathbf{x},\lambda)\,.
\end{aligned}$$

The right hand side clearly tends to zero as $\tau \to 0$. Thus, for all $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ and $\rho \in \mathrm{C}_c^1(\mathbb{R})$ we have $\lim_{E\ni\tau\to 0} I(\tau) = I(0)$, i.e.

$$\lim_{E\ni\tau\to 0} \int_{\mathbb{R}^{d+1}} h(\tau,\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda = \int_{\mathbb{R}^{d+1}} h_0(\mathbf{x},\lambda)\rho(\lambda)\phi(\mathbf{x})\,d\mathbf{x}d\lambda\,.$$

Having in mind that $h(\tau,\cdot,\cdot)$, $\tau \in E$, is bounded, and that $C_c^\infty(\mathbb{R}^{d+1})$ is dense in $\mathrm{L}^1(\mathbb{R}^{d+1})$, we complete the proof. ❑

Remark 32. If u is a bounded quasi-solution to (44), then in [47, Corollary 2.2] was proved that it admits the weak trace. The same conclusion can be derived from the previous proposition.

Indeed, let $M > 0$ be such that

$$|u(t,\mathbf{x})| \leq M\,, \quad \text{a.e } (t,\mathbf{x}) \in \mathbb{R}^{d+1}_+\,.$$

Then, by the definition of h (it is the sign function; see (48))

$$\int_{-M}^{M} h(t,\mathbf{x},\lambda)\,d\lambda = \int_{-M}^{M} \operatorname{sgn}\big(u(t,\mathbf{x}) - \lambda\big)\,d\lambda = \int_{-M}^{u(t,\mathbf{x})} d\lambda - \int_{u(t,\mathbf{x})}^{M} d\lambda = 2u(t,\mathbf{x})\,.$$

Thus, the claim follows by Proposition 31 by noting that the characteristic function $\chi_{[-M,M]}$ of the interval $[-M, M]$ is in $\mathrm{L}^1(\mathbb{R})$. More precisely, we have

$$u(t,\cdot) \overset{\star}{\rightharpoonup} u_0 := \frac{1}{2}\int_{-M}^{M} h_0(\cdot,\lambda)\,d\lambda \tag{59}$$

weakly-$\star$ in $\mathrm{L}^\infty(\mathbb{R}^d)$ as $t \to 0$, $t \in E$.

Moreover, for $\lambda \mapsto \lambda\chi_{[-M,M]}(\lambda) \in \mathrm{L}^1(\mathbb{R})$ we have

$$\int_{-M}^{M} \lambda h(t,\mathbf{x},\lambda)\,d\lambda = \int_{-M}^{u(t,\mathbf{x})} \lambda\,d\lambda - \int_{u(t,\mathbf{x})}^{M} \lambda\,d\lambda = u(t,\mathbf{x})^2 - M^2\,.$$

Therefore, one can similarly conclude that

$$u^2(t,\cdot) \stackrel{\star}{\rightharpoonup} u_1 := \int_{-M}^{M} \lambda h_0(\cdot,\lambda)\, d\lambda + M^2 \tag{60}$$

weakly-$\star$ in $\mathrm{L}^\infty(\mathbb{R}^d)$ as $t \to 0$, $t \in E$.

If one can get that $u_1 = u_0^2$, by the standard procedure a strong convergence (i.e. a strong trace) can be obtained from the above weak convergences. Namely, for an arbitrary $\varphi \in C_c(\mathbb{R}^d)$ by (59)–(60) we have

$$\begin{aligned}
\lim_{E\ni t\to 0} \int_{\mathbb{R}^d} \Big(u(t,\mathbf{x}) - u_0(\mathbf{x})\Big)^2 \varphi(\mathbf{x})\, d\mathbf{x} \\
&= \lim_{E\ni t\to 0} \int_{\mathbb{R}^d} \Big(u(t,\mathbf{x})^2 - 2u(t,\mathbf{x})u_0(\mathbf{x}) + u_0(\mathbf{x})^2\Big)\varphi(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^d} \Big(u_1(\mathbf{x}) - u_0(\mathbf{x})^2\Big)\varphi(\mathbf{x})\, d\mathbf{x}\,.
\end{aligned} \tag{61}$$

A sufficient condition for $u_1 = u_0^2$ in terms of a certain strong convergence of rescaled (sub)sequences is given in Proposition 33.

Now we use the rescaling procedure (or the so-called blow-up method) in order to obtain a sufficient condition for the existence of the strong trace. More precisely, let us change the variables in (49) in the following way: $t = \frac{\hat{t}}{m}$, $x_1 = y_1 + \frac{\hat{x}_1}{\sqrt{m}}$, $x_2 = y_2 + \frac{\hat{x}_2}{\sqrt{m}}, \dots, x_d = y_d + \frac{\hat{x}_d}{\sqrt{m}}$, i.e.

$$(t,\mathbf{x},\lambda) = \Big(\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{\sqrt{m}} + \mathbf{y}, \lambda\Big)\,, \tag{62}$$

where $\mathbf{y} \in \mathbb{R}^d$ is a fixed vector. We get that a rescaled solution to (49), denoted by

$$h_m(\hat{t},\hat{\mathbf{x}},\mathbf{y},\lambda) := h\Big(\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{m} + \mathbf{y}, \lambda\Big)$$

satisfies

$$L_{h_m} := \left(\partial_{\hat{t}} h_m + \frac{1}{\sqrt{m}} \sum_{k=1}^d \partial_{\hat{x}_k}\left(f_k h_m\right)\right) - \sum_{k,j=1}^d \partial^2_{\hat{x}_j \hat{x}_k}(a_{jk} h_m) = \frac{1}{m} \partial_\lambda \gamma_m^{\mathbf{y}}\,, \tag{63}$$

$$h_m\big|_{t=0} = h_0\big(\frac{\hat{\mathbf{x}}}{m} + \mathbf{y}, \lambda\big)\,, \tag{64}$$

where the initial conditions are understood in the weak sense, and h_0 is the weak trace from Proposition 31.

Let us remark that the equality between γ and $\gamma_m^{\mathbf{y}}$ is understood in the sense of distributions:

$$\langle \gamma_m^{\mathbf{y}}, \varphi\rangle = m^{d/2+1} \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} \varphi\big(m\,t, m\sqrt{m}(\mathbf{x}-\mathbf{y}), \lambda\big) d\gamma(t,\mathbf{x},\lambda)\,. \tag{65}$$

If we prove that the sequence

$$\int_{\mathbb{R}} h\Big(\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{\sqrt{m}} + \mathbf{y}, \lambda\Big)\rho(\lambda) d\lambda\,, \quad m \in \mathbb{N}\,, \tag{66}$$

converges strongly in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^d)$ along a subsequence, we will obtain that function u admits the trace in the sense of Definition 29. More precisely, the following proposition holds.

Proposition 33. *Let u be a bounded quasi-solution to* (44) *and let h be given by* (48)*. Assume that for every $\rho \in \mathrm{C}^1_c(\mathbb{R})$ the sequence given by* (66) *converges toward $\int_{\mathbb{R}} h_0(\mathbf{y},\lambda)\rho(\lambda)d\lambda$ in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^d_{\mathbf{y}})$ along a subsequence, where h_0 is the weak trace of h (see Proposition 31).*

Then, the function u admits the strong trace at $t=0$ and it is equal to

$$u_0(\mathbf{x}) := \frac{1}{2}\int_{-M}^{M} h_0(\mathbf{x},\lambda)\,d\lambda\,,$$

where $M = \|u\|_{\mathrm{L}^\infty(\mathbb{R}^{d+1}_+)}$.

Proof: Since both h and h_0 are bounded, using the density arguments we conclude that if the (sub)sequence from (66) converges in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+\times\mathbb{R}^d)$ for any $\rho\in \mathrm{C}^1_c(\mathbb{R})$, then it will also converge for any $\rho\in\mathrm{L}^1(\mathbb{R})$. Let us take $\rho=\chi_{[-M,M]}$, where $\chi_{[-M,M]}$ is the characteristic function of the interval $[-M,M]$, and $M>0$ is such that

$$|u(t,\mathbf{x})|\le M\,,\quad \text{a.e. } (t,\mathbf{x})\in\mathbb{R}^{d+1}_+\,.$$

Thus, for any non-negative $\varphi\in C_c(\mathbb{R}^{d+1}_+\times\mathbb{R}^d)$, it holds (along the subsequence from the formulation of the proposition)

$$\lim_{m\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^{d+1}_+}\varphi(\hat t,\hat{\mathbf{x}},\mathbf{y})\Big|\int_{-M}^{M}\Big(h\Big(\frac{\hat t}{m},\frac{\hat{\mathbf{x}}}{\sqrt{m}}+\mathbf{y},\lambda\Big)-h_0(\mathbf{y},\lambda)\Big)\,d\lambda\Big|d\mathbf{y}d\hat{\mathbf{x}}d\hat t=0\,.$$

Using the definition of the function h (it is the sign function; see (48)) we have

$$\begin{aligned}\int_{-M}^{M} h\Big(\frac{\hat t}{m},\frac{\hat{\mathbf{x}}}{\sqrt{m}}+\mathbf{y},\lambda\Big)\,d\lambda &= \int_{-M}^{M}\mathrm{sgn}\Big(u\Big(\frac{\hat t}{m},\frac{\hat{\mathbf{x}}}{\sqrt{m}}+\mathbf{y}\Big)-\lambda\Big)d\lambda\\ &= \int_{-M}^{u(\frac{\hat t}{m},\frac{\hat{\mathbf{x}}}{\sqrt{m}}+\mathbf{y})}d\lambda-\int_{u(\frac{\hat t}{m^\beta},\frac{\hat{\mathbf{x}}}{m^\alpha}+\mathbf{y})}^{M}d\lambda\\ &= 2u\Big(\frac{\hat t}{m},\frac{\hat{\mathbf{x}}}{\sqrt{m}}+\mathbf{y}\Big)\,.\end{aligned}$$

Taking this into account and the change of variables $\mathbf{z}=\frac{\hat{\mathbf{x}}}{m^\alpha}+\mathbf{y}$ (with respect to $\mathbf{y}$), the previous limit reads

$$\lim_{m\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^{d+1}_+}\varphi\Big(\frac{\hat t}{m},\hat{\mathbf{x}},\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}}\Big)\Big|2u\Big(\frac{\hat t}{m},\mathbf{z}\Big)-\int_{-M}^{M}h_0\Big(\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}},\lambda\Big)\,d\lambda\Big|d\mathbf{z}d\hat{\mathbf{x}}d\hat t=0\,.$$

Furthermore, the limit still holds if we replace $\varphi(\hat t,\hat{\mathbf{x}},\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}})$ by $\varphi(\hat t,\hat{\mathbf{x}},\mathbf{z})$ and $h_0(\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}},\lambda)$ by $h_0(\mathbf{z},\lambda)$, i.e.

$$\lim_{m\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^{d+1}_+}\varphi(\hat t,\hat{\mathbf{x}},\mathbf{z})\Big|2u\Big(\frac{\hat t}{m},\mathbf{z}\Big)-\int_{-M}^{M}h_0(\mathbf{z},\lambda)\,d\lambda\Big|d\mathbf{z}d\hat{\mathbf{x}}d\hat t=0\,. \tag{67}$$

Indeed, the first replacement is justified since

$$(\hat t,\hat{\mathbf{x}},\mathbf{z})\mapsto\Big|2u\Big(\frac{\hat t}{m},\mathbf{z}\Big)-\int_{-M}^{M}h_0\Big(\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}},\lambda\Big)\,d\lambda\Big|$$

is bounded and φ is a continuous function with compact support, hence the convergence

$$\lim_{m\to\infty}\varphi\Big(\hat t,\hat{\mathbf{x}},\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}}\Big)=\varphi(\hat t,\hat{\mathbf{x}},\mathbf{z})$$

is uniform in $(\hat t,\hat{\mathbf{x}},\mathbf{z})$. The second one follows by the convergence (implied by the continuity of the average and the Lebesgue dominated convergence theorem)

$$\lim_{m\to\infty}\int_{-M}^{M}\Big|h_0\Big(\mathbf{z}-\frac{\hat{\mathbf{x}}}{\sqrt{m}},\lambda\Big)-h_0(\mathbf{z},\lambda)\Big|\,d\lambda=0$$

in $\mathrm{L}^1_{loc}(\mathbb{R}^d\times\mathbb{R}^{d+1}_+)$.

Therefore, due to arbitrariness of φ in (67), we conclude

$$u\Big(\frac{\hat t}{m^\beta},\mathbf{z}\Big)\to \frac{1}{2}\int_{-M}^{M} h_0(\mathbf{z},\lambda)\,d\lambda,\quad m\to\infty$$

in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$ along the subsequence from the formulation of the proposition. This means that (for another subsequence not relabelled) there exists $\hat E\subseteq\mathbb{R}^+$ of full measure such that for any $\hat t\in\hat E$ we have

$$u\Big(\frac{\hat t}{m},\mathbf{z}\Big)\to \frac{1}{2}\int_{-M}^{M} h_0(\mathbf{z},\lambda)\,d\lambda = u_0(\mathbf{z}),\quad m\to\infty \tag{68}$$

in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$. It is easy to see that for E given by (56) set $E_\infty := \bigcap_{m\in\mathbb{N}} mE$ is of full measure (since E is of full measure). Thus, the intersection $\hat E\cap E_\infty$ is non-empty (in fact it is a set of full measure as well), so we can choose $\hat t\in\mathbb{R}^+$ such that (68) holds and $\frac{\hat t}{m}\in E$, $m\in\mathbb{N}$.

Now, choose $\rho(\lambda)=\lambda\,\chi_{[-M,M]}(\lambda)$ where $\chi_{[-M,M]}(\lambda)$ is the characteristic function of the interval $[-M,M]$. It holds according to Proposition 31 (see also Remark 32)

$$u^2(t,\mathbf{x})=\int_{-M}^{M}\lambda h(t,\mathbf{x},\lambda)\,d\lambda+M^2 \overset{\star}{\rightharpoonup} \int_{-M}^{M}\lambda h_0(\mathbf{x},\lambda)\,d\lambda+M^2 =: u_1(\mathbf{x})$$

in $\mathrm{L}^\infty(\mathbb{R}^d)$ as $E\ni t\to 0$. Since the weak-$\star$ convergence in $\mathrm{L}^\infty(\mathbb{R}^d)$ implies the weak convergence in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$, and since weak and strong limits coincide, from here and (68) we see that it must be $u_1=u_0^2$. Finally, by (61) (see Remark 32) we have

$$u(t,\cdot)\to u_0$$

in $\mathrm{L}^2_{loc}(\mathbb{R}^d)$ as $t\to 0$, $t\in E$, which implies the convergence in L^1_{loc}. Hence, u_0 is the strong trace. ❑

Having the last proposition in mind, we clearly need the following theorem.

Theorem 34. *Under assumption of Theorem 30, let h be given by* (48)*, and let h_0 be the weak trace of h (see Proposition 31).*

Then, for any $\rho\in\mathrm{C}^1_c(\mathbb{R})$, the sequence of functions

$$(t,\mathbf{x},\mathbf{y})\mapsto\int_{\mathbb{R}} h\Big(\frac{t}{m},\frac{\mathbf{x}}{\sqrt{m}}+\mathbf{y},\lambda\Big)\rho(\lambda)\,d\lambda$$

converges to $\int_{\mathbb{R}} h_0(\mathbf{y},\lambda)\rho(\lambda)d\lambda$ in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+\times\mathbb{R}^d)$.

Proof: First, notice that for every $\mathbf{y}\in\mathbb{R}^d$, the sequence of function (h_m) satisfies diffusive transport equation (63) which can be rewritten in the form

$$\partial_{\hat t}h_m-\sum_{k,j=1}^{d}\partial^2_{\hat x_j\hat x_k}(a_{jk}h_m)=-\frac{1}{\sqrt{m}}\sum_{k=1}^{d}\partial_{\hat x_k}(f_kh_m)+\frac{1}{m}\partial_\lambda\gamma^{\mathbf{y}}_m$$

Clearly, $\frac{1}{\sqrt{m}}\sum_{k=1}^{d}\partial_{\hat x_k}(f_kh_m)$ converges strongly in $\mathrm{L}^2_{loc}(\mathbb{R};H^{-1}_{loc}(\mathbb{R}^{d+1}_+))$. Moreover, $\frac{1}{m}\gamma^{\mathbf{y}}_m$ converges to zero in $\mathcal{M}_{loc}(\mathbb{R}^{d+1}_+\times\mathbb{R})$ (this is proved in the same way as [59, Lemma 2] or [48, Lemma 3.2]). Therefore, keeping in mind conditions (55), we can apply Theorem 1 to conclude that for every $\rho\in\mathrm{C}^1_c(\mathbb{R})$ there exists a subsequence of $(\int h_m(\cdot,\lambda)\rho(\lambda)d\lambda)$ (not relabelled) strongly converging in $\mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+)$ toward say $\int\tilde h(t,\mathbf{x},\mathbf{y},\lambda)\rho(\lambda)d\lambda$. We note that $\tilde h(t,\mathbf{x},\mathbf{y},\lambda)$ does not depend on ρ since it is a weak limit of (h_m) in $L^2_{loc}(\mathbb{R}^{d+1}_+\times\mathbb{R})$ along appropriate subsequence.

On the other hand, the sequence (h_m) satisfies (63), (64) and thus $\tilde h$ satisfies the Cauchy problem

$$\partial_{\hat t}\tilde h - \sum_{k,j=1}^{d} \partial^2_{\hat x_j \hat x_k}(a_{jk}\tilde h) = 0 \tag{69}$$

$$\tilde h\Big|_{t=0} = h_0(\mathbf{y},\lambda), \tag{70}$$

which implies $\tilde h \equiv h_0(\mathbf{y},\lambda)$ since the solution of the latter Cauchy problem is unique. Moreover, from here it follows that the entire sequence $(\int h_m(\cdot,\lambda)\rho(\lambda)d\lambda)$ must converge toward $(\int h_0(\cdot,\lambda)\rho(\lambda)d\lambda)$ (since every converging subsequence must converge toward the solution to (69), (70)).

We have thus proved that (h_m) satisfies conditions of Proposition 33 and this in turn implies that the quasi-solution to (44) indeed admits existence of strong traces. ❑

7. Appendix

In this section, we provide some auxiliary statements that we use in the proof of the velocity averaging result as well as the proof of Lemma 10.

Lemma 35. *Assume $d \geq 2$ and that a matrix function a satisfies conditions of Lemma 9. Then*

(i) *For a.e. $\lambda \in S$ and $r \in (1,d)$ the Fourier multiplier operator $\mathcal{A}_{\frac{1}{|\cdot|+\langle a(\lambda)\cdot\,|\,\cdot\rangle}}$ is bounded operator from $\mathrm{L}^r(\mathbb{R}^d)$ to $\mathrm{L}^{r^*}(\mathbb{R}^d)$ uniformly with respect to $\lambda \in S$, where $r^* = \frac{dr}{d-r}$.*

(ii) *For a.e. $\lambda \in S$ and any $k \in \{1,2,\dots,d\}$, $r \in (1,\infty)$, the operator $\partial_{x_k}\mathcal{A}_{\frac{1}{|\cdot|+\langle a(\lambda)\cdot\,|\,\cdot\rangle}}$ is continuous operator from $\mathrm{L}^r(\mathbb{R}^d)$ to $\mathrm{L}^r(\mathbb{R}^d)$ uniformly with respect to $\lambda \in S$.*

Proof: (i) It is well known that the Riesz potential $\mathcal{A}_{\frac{1}{|\boldsymbol{\xi}|}}$ is continuous mapping from $\mathrm{L}^r(\mathbb{R}^d)$ to $\mathrm{L}^{r^*}(\mathbb{R}^d)$ for $r^* = \frac{dr}{d-r}$ [54, Section 5]. Moreover, the operator $\mathcal{A}_{\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}$ is continuous operator from $\mathrm{L}^r(\mathbb{R}^d)$ to $\mathrm{L}^r(\mathbb{R}^d)$ by Lemma 10 (taking into account the invariance of multipliers under orthogonal transformations; see the proof of Lemma 9). Now, the statement follows by:

$$\mathcal{A}_{\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} = \mathcal{A}_{\frac{1}{|\boldsymbol{\xi}|}} \circ \mathcal{A}_{\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}.$$

(ii) The second part follows by applying Lemma 9 on $\psi(\boldsymbol{\xi}) = 2\pi i \xi_k$ since

$$\partial_{x_k}\mathcal{A}_{\frac{1}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}} = \mathcal{A}_{\frac{2\pi i\xi_k}{|\boldsymbol{\xi}|+\langle a(\lambda)\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\rangle}}.$$ ❑

Now, we are going to prove Lemma 10, but first we develop two auxiliary results that are given in the following two lemmata.

Lemma 36. *Let $\boldsymbol{\kappa} = (\kappa_1,\kappa_2,\dots,\kappa_d) \in [0,\infty)^d$ and define $f : \mathbb{R}^d \to \mathbb{R}$ by*

$$f(\boldsymbol{\xi}) = \frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}| + \sum_{j=1}^{d}\kappa_j\xi_j^2}.$$

For every multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$, it holds

$$(\partial^{\boldsymbol{\alpha}} f)(\boldsymbol{\xi}) = \Big(|\boldsymbol{\xi}| + \sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-1} P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big),$$

where $P_{\boldsymbol{\alpha}}(\boldsymbol{\kappa},\boldsymbol{\xi},\eta)$ is a polynomial consisting of the terms $C\boldsymbol{\kappa}^{\boldsymbol{\beta}}\boldsymbol{\xi}^{\boldsymbol{\gamma}}\eta^l$ for a constant $C = C(\boldsymbol{\alpha}, d)$, multi-indices $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathbb{N}_0^d$ and $l \in \mathbb{N}_0$, such that

$$\alpha_j + \gamma_j \geq 2\beta_j\,, \quad j = 1,\dots,d\,; \qquad |\boldsymbol{\alpha}| \geq |\boldsymbol{\beta}|\,; \qquad |\boldsymbol{\gamma}| = 1 + l + |\boldsymbol{\beta}|\,. \tag{71}$$

Proof: We prove the claim by the induction argument with respect to the order of derivative $n = |\boldsymbol{\alpha}|$.

For $n = 0$ ($|\boldsymbol{\alpha}| = 0$), we have $P_0 = |\boldsymbol{\xi}| = \frac{|\boldsymbol{\xi}|^2}{|\boldsymbol{\xi}|} = \sum\limits_{j=1}^{d} \xi_j^2 \frac{1}{|\boldsymbol{\xi}|^2}$. It is easy to check that conditions (71) are satisfied.

Assume now that the statement holds for some $n \in \mathbb{N}_0$ and let us prove that it holds for $n+1$. Let $\tilde{\boldsymbol{\alpha}} \in \mathbb{N}_0^d$, $|\tilde{\boldsymbol{\alpha}}| = n+1$, and let $s \in \{1,2,\dots,d\}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, $|\boldsymbol{\alpha}| = n$, be such that $\tilde{\boldsymbol{\alpha}} = \mathbf{e}_s + \boldsymbol{\alpha}$, where $\mathbf{e}_s$ is the s-th vector of the canonical basis of $\mathbb{R}^d$. By the Schwarz rule and the assumption of the induction argument we have

$$\begin{aligned}
\partial^{\tilde{\boldsymbol{\alpha}}} f(\boldsymbol{\xi}) &= \partial_{\xi_s}\Bigg(\Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-1} P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\Bigg)\\
&= (-|\boldsymbol{\alpha}|-1)\Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-2}\Big(\frac{\xi_s}{|\boldsymbol{\xi}|} + 2\kappa_s\xi_s\Big) P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\\
&\quad + \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|+1}(\partial_{\xi_s}P_{\boldsymbol{\alpha}})\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big) - \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|+1}(\partial_\eta P_{\boldsymbol{\alpha}})\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\frac{\xi_s}{|\boldsymbol{\xi}|^3}\\
&= \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\tilde{\boldsymbol{\alpha}}|-1}\Bigg[-|\tilde{\boldsymbol{\alpha}}|\Big(\frac{\xi_s}{|\boldsymbol{\xi}|} + 2\kappa_s\xi_s\Big) P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\\
&\quad + \Big(\frac{|\boldsymbol{\xi}|^2}{|\boldsymbol{\xi}|} + \sum_{j=1}^d \kappa_j\xi_j^2\Big)(\partial_{\xi_s}P_{\boldsymbol{\alpha}})\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big) - \Big(\frac{\xi_s}{|\boldsymbol{\xi}|^2} + \sum_{j=1}^d \kappa_j\frac{\xi_s\xi_j^2}{|\boldsymbol{\xi}|^3}\Big)(\partial_\eta P_{\boldsymbol{\alpha}})\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\Bigg]\\
&=: \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\tilde{\boldsymbol{\alpha}}|-1} P_{\tilde{\boldsymbol{\alpha}}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\,.
\end{aligned}$$

From here, a direct analysis of the six terms forming $P_{\tilde{\boldsymbol{\alpha}}}(\boldsymbol{\kappa},\boldsymbol{\xi},\eta)$ provides (71). ❏

Analogously one can prove the following result.

Lemma 37. *Let $\boldsymbol{\kappa} = (\kappa_1,\kappa_2,\dots,\kappa_d) \in [0,\infty)^d$ and $m \in \{1,2,\dots,d\}$, and define $g : \mathbb{R}^d \to \mathbb{R}$ by*

$$g(\boldsymbol{\xi}) = \frac{\kappa_m\xi_m^2}{|\boldsymbol{\xi}| + \sum\limits_{j=1}^d \kappa_j\xi_j^2}\,.$$

For every multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$, it holds

$$(\partial^{\boldsymbol{\alpha}} g)(\boldsymbol{\xi}) = \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-1} P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\kappa},\boldsymbol{\xi},\frac{1}{|\boldsymbol{\xi}|}\Big)\,,$$

where $P_{\boldsymbol{\alpha}}(\boldsymbol{\kappa},\boldsymbol{\xi},\eta)$ is a polynomial consisting of the terms $C\boldsymbol{\kappa}^{\boldsymbol{\beta}}\boldsymbol{\xi}^{\boldsymbol{\gamma}}\eta^l$ for a constant $C = C(\boldsymbol{\alpha}, d, m)$, multi-indices $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathbb{N}_0^d$ and $l \in \mathbb{N}_0$, such that

$$\alpha_j + \gamma_j \geq 2\beta_j\,, \quad j = 1,\dots,d\,; \qquad |\boldsymbol{\alpha}| + 1 \geq |\boldsymbol{\beta}|\,; \qquad |\boldsymbol{\gamma}| = 1 + l + |\boldsymbol{\beta}|\,; \qquad \beta_m \geq 1\,. \tag{72}$$

Now, we can prove Lemma 10.

Proof of Lemma 10: Since the space of L^p-Fourier multipliers is an algebra, it is sufficient to prove the statement for $s \in [0,1]$. For $s = 0$ the claim trivially holds, so let us first consider $s = 1$.

We use the Marcinkiewicz theorem (Theorem 8). Let $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$. By the previous lemmata, for both functions f and g it is sufficient to estimate

$$\boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{\kappa}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\boldsymbol{\gamma}} \frac{1}{|\boldsymbol{\xi}|^l} \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j \xi_j^2 \Big)^{-|\boldsymbol{\alpha}|-1} ,$$

where $\boldsymbol{\beta}, \boldsymbol{\gamma}$ and l satisfy (71) and (72), respectively. Thus, we have

$$\left| \boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{\kappa}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\boldsymbol{\gamma}} \frac{1}{|\boldsymbol{\xi}|^l} \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j \xi_j^2 \Big)^{-|\boldsymbol{\alpha}|-1} \right| = \prod_{j=1}^d (\kappa_j \xi_j^2)^{\beta_j} \prod_{j=1}^d |\xi_j|^{\alpha_j + \gamma_j - 2\beta_j} \frac{1}{|\boldsymbol{\xi}|^l} \frac{1}{\Big(|\boldsymbol{\xi}| + \sum\limits_{j=1}^d \kappa_j \xi_j^2 \Big)^{|\boldsymbol{\alpha}|+1}}$$

$$\leq \frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|+|\boldsymbol{\gamma}|-2|\boldsymbol{\beta}|-l}}{\Big(|\boldsymbol{\xi}| + \sum\limits_{j=1}^d \kappa_j \xi_j^2 \Big)^{|\boldsymbol{\alpha}|+1-|\boldsymbol{\beta}|}} = \frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|+1-|\boldsymbol{\beta}|}}{\Big(|\boldsymbol{\xi}| + \sum\limits_{j=1}^d \kappa_j \xi_j^2 \Big)^{|\boldsymbol{\alpha}|+1-|\boldsymbol{\beta}|}} \leq 1 ,$$

where in the first inequality we have used $|\xi_j|^{\alpha_j+\gamma_j-2\beta_j} \leq |\boldsymbol{\xi}|^{\alpha_j+\gamma_j-2\beta_j}$ as $\alpha_j + \gamma_j - 2\beta_j \geq 0$ by (71) and (72), while the last inequality is trivial since $|\boldsymbol{\alpha}| + 1 - |\boldsymbol{\beta}| \geq 0$ again by (71) and (72). Therefore, by Theorem 8, f and g are L^p-multipliers for any $p \in (1, \infty)$, and the norm of the corresponding Fourier multiplier operators is independent of $\boldsymbol{\kappa}$.

For $s \in (0,1)$ the symbols are given by $h \circ \#$, where $h(x) = x^s$ and we us $\#$ to denote either f or g, i.e. $\# \in \{f, g\}$. By the the Marcinkiewicz theorem and the generalised chain rule formula (known as the Faá di Bruno formula; see e.g. [35]) it is sufficient to estimate

$$\boldsymbol{\xi}^{\boldsymbol{\alpha}} h^{(k)}(\#(\boldsymbol{\xi})) \prod_{i=1}^k \partial^{\boldsymbol{\delta}^i} \#(\boldsymbol{\xi}) ,$$

where $h^{(k)}$ represents the derivative of the k-th order, $k \in \{1, 2, \ldots, |\boldsymbol{\alpha}|\}$ and $\boldsymbol{\delta}^i \in \mathbb{N}_0^d \setminus \{0\}$ are such that $\sum_{i=1}^k \boldsymbol{\delta}^i = \boldsymbol{\alpha}$. By lemmata 36 and 37, an arbitrary summand of $\partial^{\boldsymbol{\delta}^i} \#(\boldsymbol{\xi})$ is given by (up to a constant factor)

$$\boldsymbol{\kappa}^{\boldsymbol{\beta}^i} \boldsymbol{\xi}^{\boldsymbol{\gamma}^i} \frac{1}{|\boldsymbol{\xi}|^{l_i}} \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j \xi_j^2 \Big)^{-|\boldsymbol{\delta}^i|-1} ,$$

where $\boldsymbol{\beta}^i, \boldsymbol{\gamma}^i, l_i$ satisfy either (71) or (72), with $\boldsymbol{\delta}^i$ in place of $\boldsymbol{\alpha}$. Let us define

$$\boldsymbol{\beta} := \sum_{i=1}^k \boldsymbol{\beta}^i , \qquad \boldsymbol{\gamma} := \sum_{i=1}^k \boldsymbol{\gamma}^i , \qquad l := \sum_{i=1}^k l_i .$$

Since the derivative of h of the k-th order is equal to (up to a constant factor) x^{s-k}, we are finally left to estimate

$$\boldsymbol{\xi}^{\boldsymbol{\alpha}} (\#(\boldsymbol{\xi}))^{s-k} \boldsymbol{\kappa}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\boldsymbol{\gamma}} \frac{1}{|\boldsymbol{\xi}|^l} \Big(|\boldsymbol{\xi}| + \sum_{j=1}^d \kappa_j \xi_j^2 \Big)^{-|\boldsymbol{\alpha}|-k} , \tag{73}$$

where we have used $\sum_{i=1}^k \boldsymbol{\delta}^i = \boldsymbol{\alpha}$.

Let us consider first $\# = f$. In this case, using (71), we have

$$\alpha_j + \gamma_j \geq 2\beta_j , \quad j = 1, \ldots, d ; \qquad |\boldsymbol{\alpha}| \geq |\boldsymbol{\beta}| ; \qquad |\boldsymbol{\gamma}| = k + l + |\boldsymbol{\beta}| ,$$

and (73) reads

$$\boldsymbol{\xi}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\boldsymbol{\kappa}^{\boldsymbol{\beta}}\frac{1}{|\boldsymbol{\xi}|^{l+k-s}}\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-s}.$$

With the analogous approach as in the case $s=1$, one can get that the term above is estimated by

$$\frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|+|\boldsymbol{\gamma}|-2|\boldsymbol{\beta}|-l-k+s}}{\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+s}}=\frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+s}}{\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+s}}\leq 1\,,$$

where we have used that $|\boldsymbol{\gamma}|=k+l+|\boldsymbol{\beta}|$.

In the case $\#=g$ by (72) we have

$$\alpha_j+\gamma_j\geq 2\beta_j\,,\quad j=1,\dots,d\,;\qquad |\boldsymbol{\alpha}|+k\geq|\boldsymbol{\beta}|\,;\qquad |\boldsymbol{\gamma}|=k+l+|\boldsymbol{\beta}|\,;\qquad \beta_m\geq k\,,$$

which we use in estimating (73) to get

$$\begin{aligned}\Big|\boldsymbol{\xi}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\boldsymbol{\kappa}^{\boldsymbol{\beta}}\frac{1}{(\kappa_m\xi_m)^{k-s}}\frac{1}{|\boldsymbol{\xi}|^{l}}\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{-|\boldsymbol{\alpha}|-s}\Big|&\\ \leq(\kappa_m\xi_m^2)^{\beta_m-k+s}\prod_{\substack{j=1\\ j\neq m}}^{d}(\kappa_j\xi_j^2)^{\beta_j}&\frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|+|\boldsymbol{\gamma}|-2|\boldsymbol{\beta}|-l}}{\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{|\boldsymbol{\alpha}|+s}}\\ \leq\frac{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+k}}{\Big(|\boldsymbol{\xi}|+\sum_{j=1}^{d}\kappa_j\xi_j^2\Big)^{|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+k}}\leq 1\,.&\end{aligned}$$

Thus, the statement is proven.

❑

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