An existence result for two-phase two-component flow in porous medium by the concept of the global pressure

Anja Vrbaški and Ana Žgaljić Keko

Abstract

We derive an existence result for the system that describes two-phase two-component flow in porous media written in fully equivalent global pressure formulation. Herein we present the usage of a global pressure introduced in [2], an artificial variable that allows us to decouple original equations. By rewriting equations in the terms of that new variable, we derive an existence result for weak solutions in more tractable way compared to [19].

1 Introduction

Problems of multiphase multicomponent flow in porous media appear in many applications, as in petroleum engineering, nuclear waste storage, CO_2 sequestration and many others. For example, in nuclear waste management one often meets flow of two phases, liquid and gas, that are composed of two components, water and pure hydrogen H_2. In that application it is of great importance to know the pressure of the gas phase in a host rock, to avoid overpressurization.

Multiphase multicomponent flow in porous media is usually modeled by a system of nonlinear partial differential equations that represent mass conservation of each component, combined with initial and boundary conditions. These equations are strongly coupled, and in obtaining the solution of such a system numerical simulations play a special role. In order to obtain the numerical solution and show the convergence of numerical methods, it is important to establish the existence of the solutions. Important issue in solving a system is the choice of the primary variables. In multiphase multicomponent flow this is a challenge, and variables must be chosen carefully, see [12], [7].

Previous results on the existence of weak solutions for two-phase two-component flow have been derived in [14], [15] and [19]. In these papers they used a concept of global pressure as introduced in [16] originally for the incompressible case.

The goal of this paper is to establish the existence of weak solutions for two-phase two-component flow in porous medium in a global pressure formulation that was introduced in [2] and further studied in [6], [7], [3], [4], and which is fully equivalent to the original phase equations formulation. Compared to results of [14], [15], [19], this new formulation allows us to avoid additional regularizations, which makes our proof less technical. We consider a system with diffusivity terms in both component equations. In our work we pose assumptions on data similar to those in [19], which are more realistic than the assumptions of [14], [15]. In particular, we cover more general case of Henry law and ideal gas law since we do not assume strict positivity of the gass density. In this general case, the degeneracy caused by possible vanishing of the gas pressure combined with the degeneracy due to the saturation complicates the analysis of the system.

The rest of the paper is organized as follows. In next section we describe the mathematical and physical model used in this study and formulate the assumptions on data. Then in Section 3 we present the main result of the paper on the existence of weak solutions of the problem. In Section 4 we discretize the problem with respect to time by using a small parameter δt, introduce another regularization with respect to a small parameter ε and formulate the existence result for the discretized and non-degenerate system. The proof is carried out by using a fixed point theorem and establishing uniform estimates with respect to ε. Finally, in Section 5 uniform estimates with respect to δt are derived and the limit as δt tends to zero is performed. This completes the proof of our main result.

2 Mathematical model

We consider a porous medium filled with a fluid composed of 2 phases, *liquid* and *gas*, and we consider the fluid as a mixture of two components: a *liquid component* which does not evaporate and a *low-soluble component* (such as hydrogen) which is present mostly in the gas phase. The porous medium is assumed to be rigid and in a thermal equilibrium, while the liquid component is assumed incompressible.

The liquid phase will be denoted by index l and the gas phase by g. For each phase $\sigma \in \{l, g\}$ the phase pressures p_σ, the phase saturations S_σ, the phase mass densities ρ_σ and the phase volumetric fluxes $\mathbf{q}_\sigma$ are defined. The phase volumetric fluxes are given by the *Darcy-Muskat law* (see for instance [9]) as

$$\begin{aligned} \mathbf{q}_l &= -\lambda_l \mathbb{K}(\mathbf{x}) \left(\nabla p_l - \rho_l \mathbf{g} \right), \\ \mathbf{q}_g &= -\lambda_g \mathbb{K}(\mathbf{x}) \left(\nabla p_g - \rho_g \mathbf{g} \right), \end{aligned} \tag{1}$$

where $\mathbb{K}(\mathbf{x})$ is the absolute permeability tensor, λ_σ is the $\sigma-$phase relative mobility function, and $\mathbf{g}$ is the gravity acceleration. The porous medium is saturated by the two phases: the phase saturations satisfy

$$S_l + S_g = 1. \tag{2}$$

The phase pressures are related through a given *capillary pressure law* (see [10, 22])

$$p_c(S_g) = p_g - p_l. \tag{3}$$

From definition (3) we note that p_c is a strictly decreasing function of liquid saturation, $p_c'(S_l) < 0$. We will take that $p_c(S_l = 1) = P_0$.

In this work the phase mobilities will be considered to depend on capillary pressure, not on saturation. Therefore it is valid $\lambda_g(P_0) = 0$ which corresponds to the situation $\lambda_g(S_g = 0) = 0$ if we consider the gas mobility as a function of the saturation. In this work we will not consider the case of $S_l = 0$.

As it was said before, in the gas phase we neglect the liquid component vaporization and the gas mass density depends only on the gas pressure:

$$\rho_g = \hat{\rho}_g(p_g), \tag{4}$$

where in the case of the ideal gas law we have $\hat{\rho}_g(p_g) = C_v p_g$ with $C_v = M^h/(RT)$, where M^h is molar mass of the gas component, T is the temperature and R the universal gas constant.

The liquid component will be denoted by upper index w (suggesting *water*) and the low-soluble gas component will be denoted by upper index h (suggesting *hydrogen*). In order to describe the quantity of the gas component dissolved in the liquid we introduce mass concentration ρ_l^h which gives the mass of dissolved gas component in the volume of the liquid mixture. To simplify notation we will denote ρ_l^h by u. The assumption of thermodynamic equilibrium leads to functional dependence:

$$u = \hat{u}(p_g), \tag{5}$$

if the gas phase is present. In the absence of the gas phase u must be considered as an independent variable. The function $\hat{u}$ can be taken as a linear function $u = C_h p_g$ if the Henry law is applicable, where $C_h = HM^h$ and H is the Henry law constant. We suppose that the function $p_g \mapsto \hat{u}(p_g)$ is defined and invertible on $[0, \infty)$ and therefore we can express the gas pressure as a function of $\hat{u}$,

$$p_g = \hat{p}_g(u), \tag{6}$$

where $\hat{p}_g$ is the inverse of $\hat{u}$. Since relation (6) define the gas pressure even in the case when the gas phase is nonexistent we call this variable the *gas pseudo-pressure*. Using the relation (6) we can consider gas pseudo-pressure as persistent variable, also in the case when $S_l = 1$.

For liquid density, due to hypothesis of small solubility and liquid incompressibility we may assume constant liquid component mass concentration, i.e.:

$$\rho_l^w = \rho_l^{std}, \tag{7}$$

where ρ_l^{std} is the standard liquid component mass density (a constant). The liquid mass density is then: $\rho_l = \rho_l^{std} + u$.

Finally, the mass conservation for each component leads to the following differential equations:

$$\rho_l^{std} \Phi \frac{\partial S_l}{\partial t} + \operatorname{div}\left(\rho_l^{std} \mathbf{q}_l + \mathbf{j}_l^w\right) = \mathscr{F}^w, \tag{8}$$

$$\Phi \frac{\partial}{\partial t}\left(uS_l + \rho_g S_g\right) + \operatorname{div}\left(u\mathbf{q}_l + \rho_g \mathbf{q}_g + \mathbf{j}_l^h\right) = \mathscr{F}^h, \tag{9}$$

where the phase flow velocities, $\mathbf{q}_l$ and $\mathbf{q}_g$, are given by the Darcy-Muskat law (1), $\mathscr{F}^k$ and $\mathbf{j}_l^k$, $k \in \{w, h\}$, are respectively the $k-$component source terms and the diffusive flux in the liquid phase (see equations (10)).

The diffusive fluxes in the liquid phase are given by the Fick law which can be expressed through the gradient of the mass fractions $X_l^h = u/\rho_l$ and $X_l^w = \rho_l^w/\rho_l$ as in [11] and see also [10]:

$$\mathbf{j}_l^h = -\Phi S_l D \rho_l \nabla X_l^h, \quad \mathbf{j}_l^w = -\Phi S_l D \rho_l \nabla X_l^w, \tag{10}$$

where D is a molecular diffusion coefficient of dissolved gas in the liquid phase, possibly corrected by the tortuosity factor of the porous medium (see [10]). Note that we have $X_l^h + X_l^w = 1$, leading to $\mathbf{j}_l^h + \mathbf{j}_l^w = 0$.

The source terms $\mathscr{F}^w$ and $\mathscr{F}^h$ will be taken in the usual form:

$$\mathscr{F}^w = \rho_l^{std} F_I - \rho_l^{std} S_l F_P, \quad \mathscr{F}^h = -(uS_l + \rho_g S_g) F_P, \quad F_I, F_P \geq 0, \tag{11}$$

where we have supposed, for simplicity, that only wetting phase is injected, while composition of produced fluid is not a priori known.

In the model described here the liquid phase will be always present, which means that we will actually observe the situation when $S_l > 0$, since we will consider capillary pressure to be a positive function defined on $\langle 0, 1]$. The gas phase can disappear in certain regions of the porous domain. In these regions the system can be described by the variables p_l and u, while in the two-phase regions one can select more traditional variables S_l (or p_c) and p_g, for example. Our approach will be, similarly as in [12], [19] to select p_g as persistent variable. For second persistent variable we will choose a global pressure introduced in [2], [5], [7]. In the two-phase regions we can calculate u from relation (6) and for calculating the capillary pressure we need to obtain the value of the liquid pressure $p_l(p, p_g)$. After that we will obtain the liquid saturation by inverting the capillary pressure, $S_l = p_c^{-1}(p_g - p_l)$. In the one phase region, therefore in the liquid region we calculate p_g from relation (6) and we set capillary pressure to $p_c(0) = P_0$ and the liquid saturation to $S_l = 1$, which amounts to extending the inverse of the capillary pressure by one for negative pressures, as described in [12].

Let us note that

$$\rho_l \nabla X_l^w = -\frac{\rho_l^{std}}{\rho_l} \nabla u, \quad \rho_l \nabla X_l^h = \frac{\rho_l^{std}}{\rho_l} \nabla u.$$

Let us now rewrite the equations (8) and (9) , including expressions of Darcy velocities and Fick's law as well as source terms in particular forms.

$$\Phi \frac{\partial S_l}{\partial t} - \operatorname{div}\left(\lambda_l(S_l)\mathbb{K}(\nabla p_l - \rho_l \mathbf{g}) - \Phi S_l \frac{1}{\rho_l} D \nabla u\right) + S_l F_P = F_I, \tag{12}$$

$$\Phi \frac{\partial}{\partial t}\left(uS_l + \rho_g S_g\right) - \operatorname{div}\left(u\lambda_l(S_l)\mathbb{K}\left(\nabla p_l - \rho_l \mathbf{g}\right) + \rho_g \lambda_g(S_g)\mathbb{K}\left(\nabla p_g - \rho_g \mathbf{g}\right)\right)$$
$$- \operatorname{div}\left(\Phi S_l \frac{\rho_l^{std}}{\rho_l} D \nabla u\right) + (uS_l + \rho_g S_g) F_P = 0. \tag{13}$$

For the sake of clarity let us first consider in equations (12), (13) the liquid pressure p_l and the gas pressure p_g as independent variables. The mass concentration u and the gas mass density ρ_g

are calculated from the gas pressure as $u = \hat{u}(p_g)$ and $\rho_g = \hat{\rho}_g(p_g)$. The saturation S_l is given from the phase pressures by the capillary pressure law.

In the following section we will introduce new primary variable, the global pressure, which will be used instead of the liquid pressure.

2.1 Global pressure formulation

Let us consider the saturated region and rewrite the equations (12)-(13) by summation, in the following form:

$$\Phi \frac{\partial}{\partial t}(\rho_l S_l + \rho_g S_g) + \operatorname{div}(\rho_l \mathbf{q}_l + \rho_g \mathbf{q}_g) + (\rho_l S_l + \rho_g S_g)F_p = \rho_l^{std} F_I \tag{14}$$

$$\begin{aligned} \Phi \frac{\partial}{\partial t}\left(u S_l + \rho_g S_g\right) &- \operatorname{div}\left(u \lambda_l \mathbb{K}\left(\nabla p_l - \rho_l \mathbf{g}\right) + \rho_g \lambda_g \mathbb{K}\left(\nabla p_g - \rho_g \mathbf{g}\right)\right) \\ &- \operatorname{div}\left(\Phi S_l \frac{\rho_l^{std}}{\rho_l} D \nabla u\right) + (u S_l + \rho_g S_g) F_P = 0. \end{aligned} \tag{15}$$

We denote the total flux as the sum of the components fluxes:

$$\mathbb{Q}_t = \rho_l \mathbf{q}_l + \rho_g \mathbf{q}_g = \mathbb{Q}^w + \mathbb{Q}^h. \tag{16}$$

In the equations (14)-(15) we can consider relative permeabilities as functions of P_c instead of the saturation, simply by relations $kr_w(P_c) = kr_w(S_l(P_c))$. In such a case we may always assume dependence of the coefficients on capillary pressure instead of the saturation directly. The value of the saturation in the time derivative term we can calculate from the relation $S_l = p_c^{-1}(P_c)$. When we calculate the saturation in such a way, we can not obtain $S_l = 0$. Therefore, we always have liquid phase present.

Now, we introduce the total mobility $\lambda(p_g, P_c)$, the fractional flow functions of the gas and liquid phases $f_g(p_g, P_c)$, $f_l(p_g, P_c)$, and the mean density function $\rho(p_g, P_c)$ as follows:

$$\lambda = \rho_l \lambda_l(P_c) + \rho_g \lambda_g(P_c) \tag{17}$$

$$f_g = \frac{\rho_g \lambda_g(P_c)}{\lambda(p_g, P_c)}, \quad f_l = \frac{\rho_l \lambda_l(P_c)}{\lambda(p_g, P_c)} \tag{18}$$

$$\rho = \frac{\rho_l^2 \lambda_l(P_c) + \rho_g^2 \lambda_g(P_c)}{\lambda(p_g, P_c)} \tag{19}$$

Note that $f_g + f_l = 1$.

Note, that in the regions when both phases are present total flux can be expressed as

$$\mathbb{Q}_t = -\lambda \mathbb{K}(\nabla p_l + f_g \nabla P_c - \rho \mathbf{g}), \tag{20}$$

or

$$\mathbb{Q}_t = -\lambda \mathbb{K}(\nabla p_g - f_l \nabla P_c - \rho \mathbf{g}). \tag{21}$$

In order to write the total flux in a form of the Darcy-Muskat law we need to eliminate ∇P_c from (20) and (21) in the saturated region; in the regions where $S_l = 1$ the total flow reduces to the liquid flow. This can be done by introducing a new pressure variable p, called the global pressure, and by expressing the liquid pressure as function of p: $p_l = \tilde{p}_l(p, P_c)$. This function has to satisfy (see [5])

$$\nabla \tilde{p}_l(p, P_c) + f_g \nabla P_c = \tilde{\omega}(p, P_c) \nabla p, \tag{22}$$

for some scalar function $\tilde{\omega}$. It is sufficient to solve the following Cauchy problem

$$\begin{cases} \dfrac{\partial \tilde{p}_l}{\partial P_c}(p, P_c) = -f_g(\tilde{p}_l(p, P_c) + P_c, P_c) \quad 0 < S_l < 1 \\ \tilde{p}_l(p, P_0) = p \end{cases} \tag{23}$$

for all values of the global pressure p (which has a role of a parameter). Note that we set the initial condition $\tilde{p}_l(p, P_0) = p$, $((P_0 = p_c(S_l = 1)))$ in the liquid saturated media, the global pressure equals to the pressure of the liquid phase.

Now, we need to show how to calculate the function $\tilde{\omega}(p, P_c)$. From (22) it follows that in the saturated media ($S_l \neq 1$) we have

$$\frac{\partial \tilde{p}_g}{\partial p}(p, P_c) = \frac{\partial \tilde{p}_l}{\partial p}(p, P_c) = \tilde{\omega}(p, P_c), \tag{24}$$

and therefore, the function $\tilde{\omega}(p, P_c)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \tilde{\omega}}{\partial P_c}(p, P_c) = -\partial_{p_g} f_g(\tilde{p}_l(p, P_c) + P_c, P_c)\tilde{\omega}(p, P_c) \\ \tilde{\omega}(p, P_0) = 1. \end{cases} \tag{25}$$

Now, it is not difficult to see that the solution is given by

$$\tilde{\omega}(p, P_c) = \exp\left(-\int_{P_0}^{P_c} (\upsilon_g(p, c) - \upsilon_l(p, c)) \frac{\rho_l \rho_g \lambda_l(c) \lambda_g(c)}{(\rho_l \lambda_l(c) + \rho_g \lambda_g(c))^2} \, dc \right) \tag{26}$$

where $\upsilon_g(p, c) = \frac{\rho_g'(\tilde{p}_l(p,c)+c)}{\rho_g(\tilde{p}_l(p,c)+c)}$, $\upsilon_l(s, p) = \frac{\rho_l'(\tilde{p}_g(p,S))}{\rho_l(\tilde{p}_g(p,S))}$. The function $\tilde{\omega}$ is well defined, bounded and strictly positive for $P_c \geq P_0$ and $p \in \mathbb{R}$, and therefore the application $p \mapsto \tilde{p}_l(p, P_c)$ is invertible for all $P_c \geq P_0$. There exist constants $\omega_m, \omega_M > 0$ such that $\omega_m \leq \tilde{\omega}(p, P_c) \leq \omega_M$.

The total flux can be expressed as

$$\mathbb{Q}_t = -\lambda \mathbb{K}(\tilde{\omega} \nabla p - \rho \mathbf{g}), \tag{27}$$

where the coefficients λ, $\tilde{\omega}$ and ρ are given by (17), (26) and (19). This representation uses the global pressure in both saturated and unsaturated regions.

Remark 1 *When we consider p as a parameter we observe that* $\frac{\partial \tilde{p}_l}{\partial P_c}(p,P_c) < 0$*, and consequently* $\frac{\partial \tilde{p}_g}{\partial P_c}(p,P_c) > 0$ *for* $P_c > P_0$*, therefore we can define an inverse function* $P_c(p,p_g)$ *(while treating p as a parameter) as follows*

$$P_c(p,p_g) = \begin{cases} P_0 & p_g \le p + P_0 \\ (\tilde{p}_l)^{-1}_{P_c}(p,P_c) & p_g > p + P_0 \end{cases} \tag{28}$$

Therefore in the saturated regions we can always express capillary pressure as a function of the global pressure and the gas pressure, and this can lead us to a system with two primary unknowns p_g and p. In the region where $S_l = 1, p_c(0) = P_0$, p_g represents a pseudo-pressure, artificial variable from which we calculate $u = \rho_l^h$, and the global pressure becomes liquid pressure. Taking into account that all the coefficients in equations depend on p_g and global pressure through $P_c(p,p_g)$ we can write the phase fluxes (in the case of two-phase flow) as:

$$\mathbb{Q}^w = -\Lambda^w \mathbb{K}\nabla p + a\mathbb{K}\nabla P_c + b^w \mathbb{K}\mathbf{g} - \mathbf{j}_l^h \tag{29}$$
$$\mathbb{Q}^h = -\Lambda^h \mathbb{K}\nabla p - a\mathbb{K}\nabla P_c + b^h \mathbb{K}\mathbf{g} + \mathbf{j}_l^h \tag{30}$$

where the coefficients are given as

$$\Lambda^w = \rho_l^{std}\lambda_l(P_c)\tilde{\omega}(p,P_c), \tag{31}$$
$$\Lambda^h = (u\lambda_l(P_c) + \rho_g\lambda_g(P_c))\omega(p,P_c), \tag{32}$$
$$a = \frac{\rho_l^{std}\rho_g\lambda_g(P_c)\lambda_l(P_c)}{\lambda(p,P_c)} \tag{33}$$
$$b^w = \rho_l^{std}\rho_l\lambda_l(P_c). \tag{34}$$
$$b^h = u\rho_l\lambda_l(P_c) + \rho_g^2\lambda_g(P_c). \tag{35}$$

We can take that all coefficients depend on p and p_g, through $P_c(p,p_g)$, so we set $\omega = \tilde{\omega}(p,P_c(p,p_g))$. Finally we obtain the following system of equations which relies on the global pressure variable as one unknown, and the gas pressure as another unknown:

$$\Phi\frac{\partial}{\partial t}(\rho_l S_l + \rho_g S_g) - \operatorname{div}(\lambda\mathbb{K}(\omega\nabla p - \rho\mathbf{g})) + (\rho_l S_l + \rho_g S_g)F_p = \rho_l^{std}F_I \tag{36}$$

$$\Phi\frac{\partial}{\partial t}(uS_l + \rho_g S_g) - \operatorname{div}\left(\Lambda^h\mathbb{K}\nabla p + a\mathbb{K}\nabla P_c - b^h\mathbb{K}\mathbf{g}\right) - \operatorname{div}\left(\Phi\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right) + (uS_l + \rho_g S_g)F_p = 0. \tag{37}$$

In order that equation (36)-(37) represent also a flow in the case when $S_l = 1$ (when possibly $p_c'(1)$ is not defined) we will replace gradient of capillary pressure in the following way. We introduce θ (as in [6]),

$$\theta = \beta(P_c) = \int_0^{P_c}\sqrt{\lambda_g(s)\lambda_w(s)}\,ds, \tag{38}$$

which is well defined since β is strictly increasing. Finally, introducing the function

$$A(p,p_g)=\rho_l^{std}\rho_g(p_g)\frac{\sqrt{\lambda_l(P_c)\lambda_g(P_c)}}{\lambda(p,P_c)} \tag{39}$$

we can rewrite the system (36), (37) as

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(\rho_l^{std}S_l)-\operatorname{div}(\Lambda^w(p,p_g)\mathbb{K}\nabla p)+\operatorname{div}(A(p,p_g)\mathbb{K}\nabla\theta)\\ +\operatorname{div}(b^w\mathbb{K}\mathbf{g})+\operatorname{div}\left(\Phi\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right)+\rho_l^{std}S_lF_p=\rho_l^{std}F_I,\end{aligned} \tag{40}$$

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(uS_l+\rho_gS_g)-\operatorname{div}\left(\Lambda^h(p,p_g)\mathbb{K}\nabla p+A(p,p_g)\mathbb{K}\nabla\theta-b^h\mathbb{K}\mathbf{g}\right)\\ -\operatorname{div}\left(\Phi\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right)+(uS_l+\rho_gS_g)F_p=0.\end{aligned} \tag{41}$$

Although the form of the system (40)-(41) is written similar to [6] we can also write equivalent system to avoid the usage of θ and by using only global pressure and gas pseudo pressure. This can be done in a way that follows.

Note that in the definition of the global pressure we have imposed equality

$$\rho_l\lambda_l\nabla p_l+\rho_g\lambda_g\nabla p_g=\omega\lambda\nabla p, \tag{42}$$

and since $\rho_l>0$ we can write

$$\lambda_l\nabla p_l=\frac{\omega\lambda}{\rho_l}\nabla p-\frac{\rho_g}{\rho_l}\lambda_g\nabla p_g \tag{43}$$

This equation is valid in case when $S_l=1$ because we have imposed that in such case the liquid pressure equals global pressure.

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(\rho_l^{std}S_l)-\operatorname{div}(\frac{\rho_l^{std}\omega\lambda}{\rho_l}\mathbb{K}\nabla p-\frac{\rho_l^{std}\rho_g\lambda_g}{\rho_l}\mathbb{K}\nabla p_g)\\ +\operatorname{div}(b^w\mathbb{K}\mathbf{g})+\operatorname{div}\left(\Phi\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right)+\rho_l^{std}S_lF_p=\rho_l^{std}F_I,\end{aligned} \tag{44}$$

$$\begin{aligned}\Phi\frac{\partial}{\partial t}(uS_l+\rho_gS_g)-\operatorname{div}\left(\frac{u}{\rho_l}\omega\lambda\mathbb{K}\nabla p+\frac{\rho_l^{std}\rho_g\lambda_g}{\rho_l}\mathbb{K}\nabla p_g-b^h\mathbb{K}\mathbf{g}\right)\\ -\operatorname{div}\left(\Phi\frac{S_l\rho_l^{std}}{\rho_l}D\nabla u\right)+(uS_l+\rho_gS_g)F_p=0.\end{aligned} \tag{45}$$

To the system (44)-(45) we add initial and boundary coditions.

Boundary conditions: let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded, Lipschitz domain with its boundary divided in two parts, $\partial\Omega = \Gamma_{inj} \cup \Gamma_{imp}$, where Γ_{inj} denotes the injection boundary, and Γ_{imp} denotes the impervious one. Let $]0,T[$ be the time interval of interest and $Q_T = \Omega \times]0,T[$. We set

$$p_g = 0, \quad p = 0 \quad \text{on } \Gamma_{inj} \times]0,T[\tag{46}$$

$$\mathbb{Q}^w \cdot \mathbf{n} = \mathbf{0}, \quad \mathbb{Q}^{\mathbf{h}} \cdot \mathbf{n} = \mathbf{0} \tag{47}$$

where $\mathbf{n}$ is the outward pointing unit normal on $\partial\Omega$.

Initial conditions are given by

$$p_g(x,0) = p_g^0(x), \quad p(x,0) = p^0(x) \quad \text{in } \Omega. \tag{48}$$

2.1.1 Fundamental equality

The regularized wetting phase flux (without gravity term) can be written as:

$$\Lambda^w \mathbb{K}\nabla p - A\mathbb{K}\nabla\theta = \lambda_l(S)\rho_l \mathbb{K}\nabla p_l, \tag{49}$$

and similarly for the regularized non wetting flux:

$$\Lambda^h \mathbb{K}\nabla p + A\mathbb{K}\nabla\theta = \lambda_g(S)\rho_g \mathbb{K}\nabla p_g. \tag{50}$$

A priori estimates that will be used in the proof of Theorem 1 are based on the following equality, which can be easily checked:

$$\rho_g\lambda_g\mathbb{K}\nabla p_g \cdot \nabla p_g + \rho_l\lambda_l\mathbb{K}\nabla p_l \cdot \nabla p_l = \lambda\omega^2\mathbb{K}\nabla P \cdot \nabla P + \frac{\rho_g\rho_l}{\lambda}\mathbb{K}\nabla\theta \cdot \nabla\theta.$$

We will use it to derive the following equality:

$$\lambda_g\mathbb{K}\nabla p_g \cdot \nabla p_g + \lambda_l\mathbb{K}\nabla p_l \cdot \nabla p_l = \frac{\lambda\omega^2}{\rho_l}\mathbb{K}\nabla p \cdot \nabla p + \frac{\rho_g}{\lambda}\mathbb{K}\nabla\theta \cdot \nabla\theta + \left(1 - \frac{\rho_g}{\rho_l}\right)\lambda_g\mathbb{K}\nabla p_g \cdot \nabla p_g. \tag{51}$$

2.2 Main assumptions

(A.1) The porosity Φ belongs to $L^\infty(\Omega)$, and there exist constants, $\phi_M \geq \phi_m > 0$, such that $\phi_m \leq \Phi(x) \leq \phi_M$ a.e. in Ω. The diffusion coefficient D belongs to $L^\infty(\Omega)$, and there exists a constant $D_0 > 0$ such that $D(x) \geq D_0$ a.e. in Ω.

(A.2) The permeability tensor $\mathbb{K}$ belongs to $(L^\infty(\Omega))^{d\times d}$, and there exist constants $k_M \geq k_m > 0$, such that for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$ it holds:

$$k_m|\xi|^2 \leq \mathbb{K}(x)\xi \cdot \xi \leq k_M|\xi|^2.$$

(A.3) Relative mobilities λ_l, λ_g are defined as $\lambda_l(S_l) = kr_l(S_l)/\mu_l$ and $\lambda_g(S_l) = kr_g(S_l)/\mu_g$ where the constants $\mu_l > 0$ and $\mu_g > 0$ are the liquid and the gas viscosities, and $kr_l(S_l)$, $kr_g(S_l)$ are the relative permeability functions, satisfying $\lambda_l, \lambda_g \in C([0,1])$, $\lambda_l(0) = 0$ and $\lambda_g(1) = 0$; the function λ_l is a non decreasing and λ_g is non increasing function of S_l. Moreover, there exist a constant $\lambda_m > 0$ such that for all $S_l \in [0,1]$

$$\lambda_m \leq \lambda_l(S_l) + \lambda_g(S_l).$$

We assume also that there exists a constant $a_l > 0$ such that for all $S_l \in [0,1]$:

$$a_l S_l^2 \leq \lambda_l(S_l). \tag{52}$$

(A.4) The capillary pressure function, $p_c \in C^1((0,1])$, is monotone decreasing function satisfying $p_c(1) = 0$ and $p_c(S_l) > 0$ for $S_l \in (0,1)$ and $p_c'(S_l) \leq -M_0 < 0$ for $S_l \in (0,1]$ and some constant $M_0 < 0$. There exists a positive constant M_{p_c}

$$\int_0^1 p_c(s)\,ds = M_{p_c} < +\infty, \quad \lim_{S_l \to 0+} S_l p_c(S_l) = 0. \tag{53}$$

The inverse functions p_c^{-1} is extended as $p_c^{-1}(\sigma) = 1$ for $\sigma \leq 0$.

(A.5) The function $\hat{u}(p_g)$ is increasing C^1 function from $[0,+\infty)$ to $[0,+\infty)$ and $\hat{u}(0) = 0$. There exist constants $u_{max} > 0$, $u_0 > 0$ and $M_g > 0$ such that for all $\sigma \geq 0$ it holds,

$$|\hat{u}(\sigma)| \leq u_{max}, \quad u_0 \leq \hat{u}'(\sigma) \leq M_g.$$

For $\sigma \leq 0$ we extend $\hat{u}(\sigma)$ as a smooth, suficiently small, bounded function having global C^1 regularity. The main low solubility assumption is that the constant M_g is sufficiently small.

(A.6) Function $\hat{\rho}_g(p_g)$ is a C^1 non decreasing function on $[0,\infty)$, and there exist constants $\rho_M > 0$ and $m_g > 0$ such that for all $p_g \geq 0$ it holds

$$0 \leq \hat{\rho}_g(p_g) \leq \rho_M, \quad m_g p_g \leq \hat{\rho}_g(p_g), \quad \hat{\rho}_g(0) = 0, \quad \int_0^1 \frac{d\sigma}{\hat{\rho}_g(\sigma)} < \infty.$$

For $\sigma \leq 0$ we set $\hat{\rho}_g(\sigma) = 0$ for all $\sigma \leq 0$.

There exists a constant $\lambda_* > 0$, such that $\lambda_* \leq \lambda := \rho_l \lambda_l + \rho_g \lambda_g$.

(A.7) $F_I, F_P \in L^2(Q_T)$ and $F_I, F_P, p_g^0 \geq 0$ a.e. in Q_T.

(A.8) There exist $C > 0$ and $\tau \in (0,1)$ such that for all $S_1, S_2 \in [0,1]$

$$C \left| \int_{S_1}^{S_2} \sqrt{\lambda_l(s)\lambda_g(s))}\,ds \right|^\tau \geq |S_1 - S_2|. \tag{54}$$

Remark 2 *The function $\hat{u}(p_g)$ from (A.5) has a physical meaning only for non negative values of the pseudo pressure p_g. Regularizations applied in Section 4.1 destroy minimum principle that enforces $p_g \geq 0$ and therefore we need to extend $\hat{u}(p_g)$ for negative values of p_g as a smooth function. This extension is arbitrary and we take it sufficiently small, such that*

$$0 < \rho_l^{std} - u_{min} \leq \rho_l = \rho_l^{std} + \hat{u}(p_g) \leq \rho_l^{std} + u_{max},$$

for some constant $0 < u_{min} < \rho_l^{std}$ and $u_{min} \leq u_{max}$. We also suppose $u_{min} \leq \rho_l^{std}(1 - 1/\sqrt{2})$.

3 Existence theorem

Let us recall that the primary variables are p and p_g. The secondary variables are the functions u, ρ_g, S_l and S_g and p_l, defined as $u = \hat{u}(p_g)$, $\rho_g = \hat{\rho}_g(p_g)$, $P_c = P_c(p, p_g)$ defined by (28), $S_l = p_c^{-1}(P_c)$ and $S_g = 1 - S_l$, $p_l = p_g - P_c$. By (A.5) and (A.6) the functions u and ρ_g are bounded and for S_l, due to (A.4), we have

$$0 < S_l \leq 1. \tag{55}$$

Variational formulation is obtained by standard arguments. Taking test functions $\varphi, \psi \in C^1([0,T], V)$ where

$$V = \{\varphi \in H^1(\Omega) \colon \varphi = 0 \text{ on } \Gamma_D\}$$

we formulate the following theorem:

Theorem 1 *Let (A.1)-(A.8) hold and assume $(p^0, p_g^0) \in L^2(\Omega) \times L^2(\Omega)$, $p_g^0 \geq 0$. Then there exist functions p and p_g satisfying*

$$p_l \in L^2(Q_T), \quad p, p_g \in L^2(0,T;V),$$
$$\Phi\partial_t(uS_l + \rho_g S_g),\ \Phi\partial_t S_l \in L^2(0,T;V'),$$

such that: for all $\varphi \in L^2(0,T;V)$

$$\int_0^T \langle \Phi \frac{\partial S_l}{\partial t}, \varphi \rangle dt + \int_{Q_T} [\frac{\omega\lambda}{\rho_l}\mathbb{K}\nabla p - \frac{\rho_g \lambda_g}{\rho_l}\mathbb{K}\nabla p_g - \Phi S_l \frac{1}{\rho_l}\hat{u}'(p_g) D \nabla p_g] \cdot \nabla\varphi dxdt$$
$$+ \int_{Q_T} S_l F_P \varphi dxdt = \int_{Q_T} F_I \varphi dxdt + \int_{Q_T} \rho_l \lambda_l \mathbb{K}\mathbf{g} \cdot \nabla\varphi dxdt; \tag{56}$$

for all $\psi \in L^2(0,T;V)$

$$\int_0^T \langle \Phi \frac{\partial}{\partial t}(uS_l + \rho_g S_g), \psi \rangle dt$$
$$+ \int_{Q_T} \left(\frac{u}{\rho_l}\omega\lambda\mathbb{K}\nabla p + \frac{\rho_l^{std}\rho_g\lambda_g}{\rho_l}\mathbb{K}\nabla p_g + \Phi S_l \frac{\rho_l^{std}}{\rho_l}\hat{u}'(p_g) D\nabla p_g \right) \cdot \nabla\psi dxdt \tag{57}$$
$$+ \int_{Q_T} (uS_l + \rho_g S_g) F_P \psi dxdt = \int_{Q_T} b^h \mathbb{K}\mathbf{g} \cdot \nabla\psi dxdt.$$

Furthermore, for all $\psi \in V$ the functions

$$t \mapsto \int_\Omega \Phi S_l \psi dx, \quad t \mapsto \int_\Omega \Phi((u-\rho_g)S_l + \rho_g)\psi dx$$

are continuous in $[0,T]$ and the initial condition is satisfied in the following sense:

$$\left(\int_\Omega \Phi S_l \psi dx\right)(0) = \int_\Omega \Phi s_0 \psi dx,$$

$$\left(\int_\Omega \Phi(uS_l + \rho_g S_g)\psi dx\right)(0) = \int_\Omega \Phi(\hat{u}(p_g^0)s_0 + \hat{\rho}_g(p_g^0)(1-s_0))\psi dx,$$

for all $\psi \in V$, where $s_0 = p_c^{-1}(P_c(p^0, p_g^0))$.

The first step in proving correctness of the proposed model for two-phase compositional flow is to show that the weak solution defined in Theorem 1 satisfies $p_g \geq 0$, if the initial and the boundary conditions satisfy corresponding inequality.

Lemma 1 *Let p and p_g be given by Theorem 1. Then $p_g \geq 0$ a.e. in Q_T.*

Lemma 1 can be proved by standard technique using test function $\varphi = \frac{1}{2}(min(\hat{u}(p_g),0))^2$ in (56) and the function $\psi = min(\hat{u}(p_g),0)$ in (57). The proof is omitted here since it will be given in the discrete case in Lemma 6.

3.1 Energy estimate

In this section we develop basic a priori estimate that will serve to prove Theorem 1. Calculations in this section are formal and will be made rigorous by discretizations and regularizations presented in next sections. Motivated by [19], we introduce the test functions

$$\varphi = p_l - N(p_g), \quad \psi = M(p_g),$$

where

$$M(p_g) = \int_0^{p_g^+} \frac{1}{\hat{\rho}_g(\sigma)} d\sigma, \quad N(p_g) = \int_0^{p_g^+} \frac{\hat{u}(\sigma)}{\hat{\rho}_g(\sigma)} d\sigma. \tag{58}$$

The functions M and N are extended by zero for negative pressures.

We set

$$\mathscr{E}(p,p_g) = S_l\left(\hat{u}(p_g)M(p_g) - N(p_g)\right) + S_g\left(\hat{\rho}_g(p_g)M(p_g) - p_g\right) - \int_0^{S_l} p_c(s)ds,$$

where the dependence of $\mathscr{E}$ on global pressure is given through $S_l(p,p_g)$.

Remark 3 *From Lemma 6 in [19] we have for $p_g \geq 0$:*

$$-M_{p_c} \leq \mathscr{E}(p,p_g) \leq C(|p_g|+1). \tag{59}$$

Note that the test functions $p_l - N(p_g)$ and $M(p_g)$ satisfy the following identity:

$$\frac{\partial S_l}{\partial t}(p_l - N(p_g)) + \frac{\partial}{\partial t}(uS_l + \rho_g S_g)M(p_g) = \frac{\partial}{\partial t}\mathscr{E}(p, p_g).$$

By using the functions M, N and $\mathscr{E}$ the following result can be proved.

Lemma 2 *Let the assumptions (A.1)-(A.8) be fulfilled and let the initial conditions p^0 and p_g^0 be such that $\mathscr{E}(p^0, p_g^0) \in L^1(\Omega)$. Then there is a constant C such that each solution of* (56)*,* (57) *satisfies:*

$$\int_{Q_T} \left\{ \lambda_l(S_l)|\nabla p_l|^2 + \lambda_g(S_l)|\nabla p_g|^2 + |\nabla u|^2 \right\} \leq C, \tag{60}$$

$$\int_{Q_T} \left\{ |\nabla p|^2 + |\nabla p_g|^2 + |\nabla u|^2 \right\} \leq C, \tag{61}$$

$$\left\| \partial_t(\Phi[uS_l + \rho_g S_g]) \right\|_{L^2(0,T;H^{-1}(\Omega))} + \|\partial_t(\Phi S_l)\|_{L^2(0,T;H^{-1}(\Omega))} \leq C. \tag{62}$$

We will prove Lemma 2 by proving it for a discretized and regularized problem in Subsection 4.2 and then inferring it by passing to the limit in regularization parameter.

4 Time discretization

In this subsection we introduce a discretization of the time derivative for the regularized system in the following way: For each positive integer M we divide $[0,T]$ into M subintervals of equal length $\delta t = T/M$. We set $t_n = n\delta t$ and $J_n = (t_{n-1}, t_n]$ for $1 \leq n \leq M$, and we denote the time difference operator by

$$\partial^{\delta t} v(t) = \frac{v(t+\delta t) - v(t)}{\delta t},$$

for any $\delta t > 0$. Also, for any Hilbert space $\mathscr{H}$ we denote

$$l_{\delta t}(\mathscr{H}) = \{v \in L^\infty(0,T;\mathscr{H}) \colon v \text{ is constant in time on each subinterval } J_n \subset [0,T]\}.$$

For $v^{\delta t} \in l_{\delta t}(\mathscr{H})$ we set $v^n = (v^{\delta t})^n = v^{\delta t}|_{J_n}$ and, therefore, we can write

$$v^{\delta t} = \sum_{n=1}^{M} v^n \chi_{(t_{n-1},t_n]}(t), \quad v^{\delta t}(0) = v^0.$$

To each function $v^{\delta t} \in l_{\delta t}(\mathscr{H})$ one can assign a piecewise linear in time function

$$\tilde{v}^{\delta t} = \sum_{n=1}^{M} \left(\frac{t_n - t}{\delta t} v^{n-1} + \frac{t - t_{n-1}}{\delta t} v^n \right) \chi_{(t_{n-1},t_n]}(t), \quad \tilde{v}^{\delta t}(0) = v^0. \tag{63}$$

Then we have

$$\partial_t \tilde{v}^{\delta t}(t) = \sum_{n=1}^{M} \frac{1}{\delta t}(v^n - v^{n-1})\chi_{(t_{n-1},t_n)}(t) = \partial^{-\delta t} v^{\delta t}(t), \quad \text{for } t \neq n\delta t, n = 0, 1, \ldots, N.$$

Finally, for any function $f \in L^1(0,T;\mathscr{H})$ we define $f^{\delta t} \in l_{\delta t}(\mathscr{H})$ by,

$$f^{\delta t}(t) = \frac{1}{\delta t}\int_{J_n} f(\tau)d\tau, \quad t \in J_n.$$

As before, we will denote by

$$u^{\delta t} = \hat{u}(p_g^{\delta t}),\ \rho_l^{\delta t} = \rho_l^{std} + u^{\delta t},\ \rho_g^{\delta t} = \hat{\rho}_g(p_g^{\delta t}),\ P_c^{\delta t} = P_c(p^{\delta t}, p_g^{\delta t}),\ S_l^{\delta t} = p_c^{-1}(P_c^{\delta t}).$$

Using notation

$$\omega^{\delta t} = \omega(p^{\delta t}, p_g^{\delta t}), \quad \lambda^{\delta t} = \lambda(p^{\delta t}, p_g^{\delta t}), \quad \lambda_g^{\delta t} = \lambda_g(S_l^{\delta t}), \quad \lambda_l^{\delta t} = \lambda_l(S_l^{\delta t}) \tag{64}$$

the discrete system is defined as follows: for given p^0, p_g^0 find $p^{\delta t} \in l_{\delta t}(V)$ and $p_g^{\delta t} \in l_{\delta t}(V)$ satisfying

$$\begin{aligned}\int_{Q_T} \Phi \partial^{-\delta t}(S_l^{\delta t})\varphi\, dx dt + \int_{Q_T} [\frac{\omega^{\delta t}\lambda^{\delta t}}{\rho_l^{\delta t}}\mathbb{K}\nabla p^{\delta t} - \frac{\rho_g^{\delta t}\lambda_g^{\delta t}}{\rho_l^{\delta t}}\mathbb{K}\nabla p_g^{\delta t} - \Phi S_l^{\delta t}\frac{1}{\rho_l^{\delta t}} D\nabla u^{\delta t}]\cdot\nabla\varphi\, dx dt \\ + \int_{Q_T} S_l^{\delta t} F_P^{\delta t}\varphi\, dx dt = \int_{Q_T} F_l^{\delta t}\varphi\, dx dt + \int_{Q_T}\rho_l^{\delta t}\lambda_l^{\delta t}\mathbb{K}\mathbf{g}\cdot\nabla\varphi\, dx dt\end{aligned} \tag{65}$$

for all $\varphi \in l_{\delta t}(V)$;

$$\begin{aligned}&\int_{Q_T} \Phi\partial^{-\delta t}\left(u^{\delta t}S_l^{\delta t} + \rho_g^{\delta t}(1 - S_l^{\delta t})\right)\psi\, dx dt \\ &+ \int_{Q_T}\left(\frac{u^{\delta t}}{\rho_l^{\delta t}}\omega^{\delta t}\lambda^{\delta t}\mathbb{K}\nabla p^{\delta t} + \frac{\rho_l^{std}}{\rho_l^{\delta t}}\rho_g^{\delta t}\lambda_g^{\delta t}\mathbb{K}\nabla p_g^{\delta t} + \Phi S_l^{\delta t}\frac{\rho_l^{std}}{\rho_l^{\delta t}}D\nabla u^{\delta t}\right)\cdot\nabla\psi\, dx dt \\ &+ \int_{Q_T}(u^{\delta} S_l^{\delta t} + \rho_g^{\delta t}S_g^{\delta t})F_P^{\delta t}\psi\, dx dt \\ &= \int_{Q_T}\left(\rho_l^{\delta t}u^{\delta t}\lambda_l^{\delta t} + (\rho_g^{\delta t})^2\lambda_g^{\delta t}\right)\mathbb{K}\mathbf{g}\cdot\nabla\psi dx dt\end{aligned} \tag{66}$$

for all $\psi \in l_{\delta t}(V)$. For $t \le 0$ we set $p^{\delta t} = p^0$, $p_g^{\delta t} = p_g^0$.

Theorem 2 *Assume* $(A.1) - (A.8)$, p_g^0, $p^0 \in L^2(\Omega)$ *and* $p_g^0 \ge 0$. *Then for all* δt *exist functions* $p^{\delta t}, p_g^{\delta t} \in l_{\delta t}(V)$ *satisfying (65), (66), such that*

$$p_g^{\delta t} \ge 0 \quad a.e.\ in\ Q_T.$$

4.1 Application of the Schauder's fixed point theorem. Proof of Proposition 1. Proof of Theorem 2

The proof of Theorem 2 is based on the Schauder fixed point theorem. It is enough to prove that for given $p^{k-1}, p_g^{k-1} \in V$ the problem (67)-(68) has a solution p^k, p_g^k.

Let us fix $1 \leq k \leq M$. We need to establish the existence of functions $p^k, p_g^k \in V$ that satisfy

$$\frac{1}{\delta t}\int_\Omega \Phi(S_l^k - S_l^{k-1})\varphi\,dx + \int_\Omega [\frac{\omega^k\lambda^k}{\rho_l^k}\mathbb{K}\nabla p^k - \frac{\rho_g^k\lambda_g^k}{\rho_l^k}\mathbb{K}\nabla p_g^k - \Phi S_l^k\frac{1}{\rho_l^k}D\nabla u^k]\cdot\nabla\varphi\,dx$$
$$+\int_\Omega S_l^k F_P^k\varphi\,dx = \int_\Omega F_l^k\varphi\,dx + \int_\Omega \rho_l^k\lambda_l^k\mathbb{K}\mathbf{g}\cdot\nabla\varphi\,dx \tag{67}$$

for all $\varphi \in V$ and

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\left(u^kS_l^k + \rho_g^k(1-S_l^k)\right) - \left(u^{k-1}S_l^{k-1} + \rho_g^{k-1}(1-S_l^{k-1})\right)\right)\psi\,dx$$
$$+\int_\Omega \left(\frac{u^k}{\rho_l^k}\omega^k\lambda^k\mathbb{K}\nabla p^k + \frac{\rho_l^{std}}{\rho_l^k}\rho_g^k\lambda_g^k\mathbb{K}\nabla p_g^k + \Phi S_l^k\frac{\rho_l^{std}}{\rho_l^k}D\nabla u^k\right)\cdot\nabla\psi\,dx$$
$$+\int_\Omega (u^kS_l^k + \rho_g^kS_g^k)F_P^k\psi\,dx = \int_\Omega \left(\rho_l^ku^k\lambda_l^k + (\rho_g^k)^2\lambda_g^k\right)\mathbb{K}\mathbf{g}\cdot\nabla\psi dx \tag{68}$$

for all $\psi \in V$. Here, as always we use notation:

$$u^k = \hat{u}(p_g^k),\ \rho_l^k = \rho_l^{std} + u^k,\ \rho_g^k = \hat{\rho}_g(p_g^k),\ P_c^k = P_c(p^k, p_g^k),\ S_l^k = p_c^{-1}(P_c^k).$$

and we denote:

$$\omega^k = \omega(p^k, p_g^k), \quad \lambda^k = \lambda(p^k, p_g^k), \quad \lambda_g^k = \lambda_g(S_l^k), \quad \lambda_l^k = \lambda_l(S_l^k). \tag{69}$$

From now on, in order to simplify the notation we will omit the superscript k, and will assume k is being fixed. In order to apply Schauder's fixed point theorem, we will add a regularization of the system: we introduce a small parameter ε, and replace $\hat{\rho}_g(p_g)$, $\lambda_l(S_l)$ and $\lambda_g(S_l)$ by $\hat{\rho}_g^\varepsilon(p_g) := \hat{\rho}_g(p_g) + \varepsilon$, $\lambda_l^\varepsilon(S_l) := \lambda_l(S_l) + \varepsilon$ and $\lambda_g^\varepsilon := \lambda_g(S_l) + \varepsilon$, respectively.

In order to simplify further notation we will denote function S_l by S in the rest of the section. Let p^* and p_g^* be given functions from $L^2(\Omega)$. We denote

$$P_c^* = P_c(p^*, p_g^*),\ S^* = p_c^{-1}(P_c^*),\ u^* = \hat{u}(p_g^*),\ \rho_g^{\varepsilon,*} = \hat{\rho}_g^\varepsilon(p_g^*),\ \rho_l^* = \rho_l^{std} + u^*.$$

Let us rewrite previous equations, with regularized coefficients, in simplified notation: find $p, p_g \in V$ such that

$$\frac{1}{\delta t}\int_\Omega \Phi(S - S^*)\varphi\,dx + \int_\Omega [\frac{\omega\lambda}{\rho_l}\mathbb{K}\nabla p - \frac{\rho_g^\varepsilon\lambda_g^\varepsilon}{\rho_l}\mathbb{K}\nabla p_g - \Phi S\frac{1}{\rho_l}D\nabla u]\cdot\nabla\varphi\,dx$$
$$+\int_\Omega SF_P\varphi\,dx = \int_\Omega F_l\varphi\,dx + \int_\Omega \rho_l\lambda_l\mathbb{K}\mathbf{g}\cdot\nabla\varphi\,dx \tag{70}$$

for all $\varphi \in V$ and

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\left(uS + \rho_g^\varepsilon(1-S)\right) - \left(u^*S^* + \rho_g^{\varepsilon,*}(1-S^*)\right)\right)\psi\,dx$$
$$+\int_\Omega \left(\frac{u}{\rho_l}\omega\lambda\mathbb{K}\nabla p + \frac{\rho_l^{std}}{\rho_l}\rho_g^\varepsilon\lambda_g^\varepsilon\mathbb{K}\nabla p_g + \Phi S\frac{\rho_l^{std}}{\rho_l}D\nabla u\right)\cdot\nabla\psi\,dx$$
$$+\int_\Omega (uS + \rho_g^\varepsilon(1-S))F_P\psi\,dx = \int_\Omega \left(\rho_lu\lambda_l(S) + (\rho_g^\varepsilon)^2\lambda_g(S)\right)\mathbb{K}\mathbf{g}\cdot\nabla\psi dx \tag{71}$$

for all $\psi \in V$.

By multiplying (70) with ρ_l^{std} and summing it with (71) we get

$$\begin{aligned}
&\frac{1}{\delta t}\int_\Omega \Phi((\rho_l S+\rho_g^\varepsilon(1-S))-(\rho_l^* S^*+\rho_g^{\varepsilon,*}(1-S^*)))\varphi\,dx+\int_\Omega \omega\lambda\mathbb{K}\nabla p\cdot\nabla\varphi\,dx\\
&\quad+\int_\Omega(\rho_l S+\rho_g^\varepsilon(1-S))F_P\varphi\,dx \qquad (72)\\
&\quad=\int_\Omega \rho_l^{std}F_l\varphi\,dx+\int_\Omega\left(\rho_l^2\lambda_l(S)+(\rho_g^\varepsilon)^2\lambda_g(S)\right)\mathbb{K}\mathbf{g}\cdot\nabla\varphi\,dx.
\end{aligned}$$

Note that the unknowns p, p_g in equations (70)–(72) depend on parameter ε. For simplicity of notation, this dependancy will be omitted in writing until we pass to the limite as $\varepsilon \to 0$ in Subsection 4.2.

The system (71)-(72) is equivalent to the system (70)-(71). We will now consider the system (71)-(72).

The existence of the solution (p, p_g) for the system (71)–(72) will be proved by fixed point theorem. This technique is common and is used in [1], [20] and similar papers. We will apply the following formulation of the Schauder's fixed point theorem.

Theorem 3 *Let K be a nonempty, compact, convex subset of a Banach space X, and suppose $T : K \mapsto K$ is a continuous operator. Then T has a fixed point.*

The aim of this subsection is therefore to prove the following proposition by using Schauder's fixed point theorem.

Proposition 1 *For given $(p^*, p_g^*) \in L^2(\Omega)\times L^2(\Omega)$ there exist $(p, p_g) \in V\times V$ that solve (71)–(72) .*

Proof.

Let us define mapping $\mathscr{T} : L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)\times L^2(\Omega)$ by $\mathscr{T}(\overline{p},\overline{p}_g) = (p, p_g)$, where (p, p_g) is a unique solution of linear system (73)–(74) below. In this system we use the following notation:

$$\begin{aligned}
&\overline{S}=p_c^{-1}(P_c(\overline{p},\overline{p}_g)),\quad \overline{u}=\hat{u}(\overline{p}_g),\quad \overline{\rho}_g^\varepsilon=\hat{\rho}_g^\varepsilon(\overline{p}_g),\quad \overline{\rho}_l=\rho_l^{std}+\overline{u};\\
&\overline{\omega}=\omega(\overline{p},\overline{p}_g),\quad \overline{\lambda_l^\varepsilon}=\lambda_l^\varepsilon(\overline{S}),\quad \overline{\lambda_g^\varepsilon}=\lambda_g^\varepsilon(\overline{S}).
\end{aligned}$$

Linearized variational problem that defines the mapping $\mathscr{T}$ is the following:

$$\begin{aligned}
&\frac{1}{\delta t}\int_\Omega \Phi((\overline{\rho}_l\overline{S}+\overline{\rho}_g^\varepsilon(1-\overline{S}))-(\rho_l^* S^*+\rho_g^{\varepsilon,*}(1-S^*)))\varphi\,dx+\int_\Omega \overline{\omega}\overline{\lambda}\mathbb{K}\nabla p\cdot\nabla\varphi\,dx\\
&\quad+\int_\Omega(\overline{\rho}_l\overline{S}+\overline{\rho}_g^\varepsilon(1-\overline{S}))F_P\varphi\,dx \qquad (73)\\
&\quad=\int_\Omega F_l\varphi\,dx+\int_\Omega\left(\overline{\rho_l}^2\overline{\lambda_l}+(\overline{\rho}_g^\varepsilon)^2\overline{\lambda_g}\right)\mathbb{K}\mathbf{g}\cdot\nabla\varphi\,dx
\end{aligned}$$

for all $\varphi \in V$ and

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\left(\overline{u}\overline{S}+\overline{\rho}_g^\varepsilon(1-\overline{S})\right)-\left(u^*S^*+\rho_g^{\varepsilon,*}(1-S^*)\right)\right)\psi\,dx$$
$$+\int_\Omega\left(\frac{\overline{u}}{\overline{\rho}_l}\overline{\omega}\overline{\lambda}\mathbb{K}\nabla p+\frac{\rho_l^{std}}{\overline{\rho}_l}\overline{\rho}_g^\varepsilon\overline{\lambda_g^\varepsilon}\mathbb{K}\nabla p_g+\Phi\overline{S}\frac{\rho_l^{std}}{\overline{\rho}_l}D\hat{u}'(\overline{p}_g)\nabla p_g\right)\cdot\nabla\psi\,dx$$
$$+\int_\Omega(\overline{u}\overline{S}+\overline{\rho}_g^\varepsilon(1-\overline{S}))F_P\psi\,dx=\int_\Omega\left(\overline{\rho}_l\overline{u}\overline{\lambda_l}+(\overline{\rho}_g^\varepsilon)^2\overline{\lambda_g}\right)\mathbb{K}\mathbf{g}\cdot\nabla\psi dx \tag{74}$$

for all $\psi \in V$.

Note that (73) is a linear elliptic equation for the function p, which can be written as

$$A_1(p,\varphi)=f_1(\varphi),$$

where

$$A_1(p,\varphi)=\int_\Omega\overline{\omega}\overline{\lambda}\mathbb{K}\nabla p\cdot\nabla\varphi\,dx$$

and $f_1(\varphi)$ is given by remaining terms in equation (73) as follows:

$$f_1(\varphi)=\int_\Omega F_I\varphi\,dx+\int_\Omega\left(\overline{\rho_l}^2\overline{\lambda_l}+(\overline{\rho}_g^\varepsilon)^2\overline{\lambda_g}\right)\mathbb{K}\mathbf{g}\cdot\nabla\varphi\,dx-\int_\Omega(\overline{\rho}_l\overline{S}+(\overline{\rho}_g^\varepsilon)(1-\overline{S}))F_P\varphi\,dx \tag{75}$$
$$-\frac{1}{\delta t}\int_\Omega\Phi((\overline{\rho}_l\overline{S}+\overline{\rho}_g^\varepsilon(1-\overline{S}))-(\rho_l^*S^*+\rho_g^{\varepsilon,*}(1-S^*)))\varphi\,dx.$$

A_1 is obviously bilinear; it is bounded by using boundedness of ω and it is coercive. Using boundedness of the densities ((A.6)); (A.3) and (A.7) one can easily prove:

$$|f_1(\varphi)|\le C\|\varphi\|_V.$$

Let us note that the constant C, as well as other constants in this proof, depends on parameter ε.

Now, the equation (73) has a unique solution by the Lax–Milgram lemma. By setting $\varphi = p$ in (73) a straightforward estimate follows :

$$\omega_m\lambda_*k_m\int_\Omega|\nabla p|^2\,dx\le C_1(1+\frac{1}{\delta t})(\|\Phi\|_{L^2(\Omega)}+\|F_I\|_{L^2(\Omega)}+\|F_P\|_{L^2(\Omega)})\|p\|_{L^2(\Omega)} \tag{76}$$
$$+C_2\|\nabla p\|_{L^2(\Omega)},$$

where C_1, C_2 are constants. Therefore, by an application of the Poincaré inequality we have

$$\|p\|_{H^1(\Omega)}\le C, \tag{77}$$

where C is independent of $\overline{p},\overline{p}_g$.

Now, since p is known, the equation(74) becomes a linear elliptic equation for the function p_g that can be written as

$$A_2(p_g,\psi)=f_2(\psi),$$

where

$$A_2(p_g,\psi)=\frac{\rho_l^{std}}{\overline{\rho}_l}\overline{\rho}_g^{\varepsilon}\left(\overline{\lambda_g}\mathbb{K}\nabla p_g+\Phi\overline{S}\frac{1}{\overline{\rho}_g^{\varepsilon}}D\hat{u}'(\overline{p}_g)\nabla p_g\cdot\right)\nabla\psi$$

and where linear functional $f_2(\psi)$ is given by the remaining terms in equation (74) as

$$f_2(\psi)=\int_\Omega\left(\overline{\rho}_l\overline{u}\overline{\lambda_l^{\varepsilon}}+(\overline{\rho}_g^{\varepsilon})^2\overline{\lambda_g^{\varepsilon}}\right)\mathbb{K}\mathbf{g}\cdot\nabla\psi dx-\int_\Omega(\overline{u}\overline{S}+\overline{\rho}_g^{\varepsilon}(1-\overline{S}))F_P\psi\,dx \tag{78}$$
$$-\frac{1}{\delta t}\int_\Omega\Phi\left(\left(\overline{u}\overline{S}+\overline{\rho}_g^{\varepsilon}(1-\overline{S})\right)-\left(u^*S^*+\rho_g^{\varepsilon,*}(1-S^*)\right)\right)\psi\,dx+\int_\Omega\frac{\overline{u}}{\overline{\rho}_l}\overline{\omega}\overline{\lambda^{\varepsilon}}\mathbb{K}\nabla p\nabla\psi.$$

We can obtain

$$A_2(p_g,p_g)\geq\frac{\rho_l^{std}}{\rho_l^{std}+u_{max}}\varepsilon\left(\frac{kr_g(\overline{S})k_m}{\mu_g}+\frac{\Phi\overline{S}D}{\rho_M+\varepsilon}u_0\right)\nabla p_g\cdot\nabla p_g.$$

By using a variant of Lemma 3 in [19] then it follows

$$A_2(p_g,p_g)\geq\varepsilon C_3\nabla p_g\cdot\nabla p_g$$

so the form $A_2(p_g,\psi)$ is coercive.

By using boundedness of the coefficients we can obtain:

$$|f_2(\psi)|\leq C\|\psi\|_V.$$

Lax–Milgram lemma now gives a unique solution $p_g\in V$ of the equation (74). This ensures that the map $\mathscr{T}$ is well defined on $L^2(\Omega)\times L^2(\Omega)$.

Furthermore, by setting $\psi=p_g$ in (74) a straightforward estimate follows for ε sufficiently small, namely for $\varepsilon<\rho^{std}$:

$$\varepsilon C_3\int_\Omega|\nabla p_g|^2\,dx\leq C_4(1+\frac{1}{\delta t})(\|\Phi\|_{L^2(\Omega)}+\|F_I\|_{L^2(\Omega)}+\|F_P\|_{L^2(\Omega)})\|p_g\|_{L^2(\Omega)} \tag{79}$$
$$+C_5\|\nabla p_g\|_{L^2(\Omega)},$$

where C_4,C_5 are constants. Therefore, by an application of the Poincaré inequality we have

$$\|p_g\|_{H^1(\Omega)}\leq C, \tag{80}$$

where C is independent of $\overline{p},\overline{p}_g$.

From the estimates (77), (80) we conclude that the mapping $\mathscr{T}$ maps $L^2(\Omega)\times L^2(\Omega)$ to a bounded set in $H^1(\Omega)\times H^1(\Omega)$ and therefore there is a ball K in $L^2(\Omega)\times L^2(\Omega)$ which $\mathscr{T}$ maps to itself. K is nonempty, convex and compact. In order to apply the Schauder 's theorem, it remains to show the continuity of the map $\mathscr{T}$. Let $(\overline{p_i},\overline{p_g^i})$ be a sequence in $L^2(\Omega)\times L^2(\Omega)$ which converges strongly to some $(\overline{p},\overline{p_g})$. We denote $(p^i,p_g^i)=\mathscr{T}(\overline{p_i},\overline{p_g^i})$. Since $(p^i,p_g^i)\in K$, and $H^1(\Omega)\times H^1(\Omega)$ is relatively compact in $L^2(\Omega)\times L^2(\Omega)$, we conclude that, up to a subsequence,

(p^i, p_g^i) converges to some (p, p_g) strongly in $L^2(\Omega)\times L^2(\Omega)$, a. e. in $L^2(\Omega)\times L^2(\Omega)$ and weakly in $H^1(\Omega)\times H^1(\Omega)$. One can pass to the limit in (73)-(74) using boundedness of the coefficients and the Lebesgue theorem, giving $(p, p_g) = \mathscr{T}(\overline{p}, \overline{p_g})$, which proves the continuity of the mapping $\mathscr{T}$ since the limit (p, p_g) is unique.

We conclude that all assumptions of the Schauder's fixed point theorem are satisfied which proves the Proposition 1, i. e. there is a solution of (71)–(72) (which depends on ε). In order to prove the existence for the system (67)-(68), we need to pass to the limit as $\varepsilon \to 0$ in (71)–(72) (or in (70)–(71)).

4.2 Passage to the limit as $\varepsilon \to 0$

Now we will denote the solution to (70)–(71) by p^ε, $p_g^\varepsilon \in V$. As before, we denote $p_l^\varepsilon = p_g^\varepsilon - P_c(p^\varepsilon, p_g^\varepsilon)$, $u^\varepsilon = \hat{u}(p_g^\varepsilon)$ and $\theta^\varepsilon = \beta(P_c(p^\varepsilon, p_g^\varepsilon))$. Let the test functions in system (70)–(71) be

$$\varphi = p_l^\varepsilon - N^\varepsilon(p_g^\varepsilon) \text{ in (70)}, \quad \psi = M^\varepsilon(p_g^\varepsilon) \text{ in (71)},$$

where

$$M^\varepsilon(p_g^\varepsilon) = \int_0^{p_g^\varepsilon} \frac{1}{\hat{\rho}_g^\varepsilon(\sigma)}\, d\sigma, \quad N^\varepsilon(p_g^\varepsilon) = \int_0^{p_g^\varepsilon} \frac{\hat{u}(\sigma)}{\hat{\rho}_g^\varepsilon(\sigma)} d\sigma. \tag{81}$$

We also define

$$\mathscr{E}^\varepsilon(p^\varepsilon, p_g^\varepsilon) = S_l^\varepsilon \left(\hat{u}(p_g^\varepsilon) M^\varepsilon(p_g^\varepsilon) - N^\varepsilon(p_g^\varepsilon)\right) + S_g^\varepsilon \left(\hat{\rho}_g^\varepsilon(p_g^\varepsilon) M^\varepsilon(p_g^\varepsilon) - p_g^\varepsilon\right) - \int_0^{S_l^\varepsilon} p_c(s) ds. \tag{82}$$

The dependence of $\mathscr{E}^\varepsilon$ on global pressure is given through $S_l^\varepsilon(p^\varepsilon, p_g^\varepsilon)$.

Remark 4 *As earlier, we note that for $p_g \geq 0$:*

$$-M_{p_c} \leq \mathscr{E}^\varepsilon(p, p_g) \leq C(|p_g| + 1).$$

Note that the test functions $p_l^\varepsilon - N(p_g^\varepsilon)$ and $M(p_g^\varepsilon)$ satisfy the following identity:

$$\frac{\partial S_l^\varepsilon}{\partial t}\left(p_l^\varepsilon - N(p_g^\varepsilon)\right) + \frac{\partial}{\partial t}\left(u^\varepsilon S_l^\varepsilon + \rho_g S_g^\varepsilon\right) M(p_g^\varepsilon) = \frac{\partial}{\partial t} \mathscr{E}^\varepsilon(p^\varepsilon, p_g^\varepsilon).$$

Following lines of proof of Lemma 8 in [19], we can prove the following result.

Lemma 3 *There is a constant C independent of δt and ε such that each solution of* (70)*,* (71) *satisfies:*

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\mathscr{E}^\varepsilon(p^\varepsilon, p_g^\varepsilon) - \mathscr{E}^\varepsilon(p^*, p_g^*)\right) dx \tag{83}$$
$$+ \int_\Omega \left(\lambda_l^\varepsilon \mathbb{K} \nabla p_l^\varepsilon \cdot \nabla p_l^\varepsilon + \lambda_g^\varepsilon \mathbb{K} \nabla p_g^\varepsilon \cdot \nabla p_g^\varepsilon + \nabla u^\varepsilon \cdot \nabla u^\varepsilon\right) dx \leq C.$$

By using (59) with $p_g^* \geq 0$, from (83) first we conclude that $(u^\varepsilon)_\varepsilon$ is uniformly bounded in V. Due to (A.5), this implies that also $(p_g^\varepsilon)_\varepsilon$ is uniformly bounded in V. Next, since we have bounds from below $\lambda_l^\varepsilon \geq \varepsilon$ and $\lambda_g^\varepsilon \geq \varepsilon$, we also get that $(\sqrt{\varepsilon}\nabla p_l^\varepsilon)_\varepsilon$ and $(\sqrt{\varepsilon}\nabla p_g^\varepsilon)_\varepsilon$ are uniformly bounded in $L^2(\Omega)$. Then we employ the equality (51) to obtain from (83):

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\mathscr{E}^\varepsilon(p^\varepsilon, p_g^\varepsilon) - \mathscr{E}^\varepsilon(p^*, p_g^*)\right) dx \tag{84}$$
$$+\int_\Omega \left(\frac{\lambda\omega^2}{\rho_l}\mathbb{K}\nabla p^\varepsilon \cdot \nabla p^\varepsilon + \frac{\hat{\rho}_g^\varepsilon}{\lambda}\mathbb{K}\nabla\theta^\varepsilon \cdot \nabla\theta^\varepsilon + \left(1 - \frac{\hat{\rho}_g^\varepsilon}{\rho_l}\right)\lambda_g \mathbb{K}\nabla p_g^\varepsilon \cdot \nabla p_g^\varepsilon\right) dx \leq C.$$

From (84) we conclude that the global pressure $(p^\varepsilon)_\varepsilon$ is uniformly bounded in V. Moreover, from the bound $\rho_g^\varepsilon \geq \varepsilon$, we also obtain that $(\sqrt{\varepsilon}\nabla\theta^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\Omega)$.

Namely, it holds

Lemma 4 *There is a constant C independent of δt and ε such that each solution of* (70), (71) *satisfies:*

$$\int_\Omega \left\{|\sqrt{\varepsilon}\nabla p_l^\varepsilon|^2 + |\sqrt{\varepsilon}\nabla p_g^\varepsilon|^2 + |\sqrt{\varepsilon}\nabla\theta^\varepsilon|^2 + |\nabla p^\varepsilon|^2 + |\nabla p_g^\varepsilon|^2 + |\nabla u^\varepsilon|^2\right\} dx \leq C. \tag{85}$$

From the above estimates, uniform in ε, the following convergencies can be deducted:

Lemma 5 *Let $(p^\varepsilon,\ p_g^\varepsilon)$ be the solution to* (67)-(68). *There there exist functions $p, p_g \in L^2(\Omega)$ such that on a subsequence as $\varepsilon \to 0$ it holds:*

$$p^\varepsilon \to p \text{ weakly in } V \text{ and a. e. in } \Omega; \tag{86}$$
$$p_g^\varepsilon \to p_g \text{ weakly in } V \text{ and a. e. in } \Omega; \tag{87}$$
$$P_c(p^\varepsilon, p_g^\varepsilon) \to P_c(p, p_g) \text{ a. e. in } \Omega; \tag{88}$$
$$S^\varepsilon = p_c^{-1}(P_c(p^\varepsilon, p_g^\varepsilon)) \to S := p_c^{-1}(P_c(p, p_g)) \text{ a. e. in } \Omega; \tag{89}$$
$$u^\varepsilon \to u := \hat{u}(p_g) \text{ weakly in } V \text{ and a. e. in } \Omega; \tag{90}$$
$$\hat{\rho}_g^\varepsilon(p_g^\varepsilon) \to \rho_g := \hat{\rho}_g(p_g) \text{ a. e. in } \Omega; \tag{91}$$
$$\rho_l(p_g^\varepsilon) \to \rho_l(p_g) := \rho_l^{std} + \hat{u}(p_g) \text{ a. e. in } \Omega. \tag{92}$$

Now we are able to pass to the limit as $\varepsilon \to 0$ in system (70)–(71) by using the convergence results of Lemma 5 and boundedness of the coefficients. In this way we have proved the existence of a unique solution of the problem (67)-(68).

In order to complete the proof of Theorem 2 we need the following result.

Lemma 6 *Let $p^{k-1}, p_g^{k-1} \in V, p_g^{k-1} \geq 0$ for fixed k. Let (p^k, p_g^k) be a solution of* (67)-(68). *Then $p_g^k \geq 0$ a.e. in Ω.*

Proof. Let us put the test function $\varphi = \frac{1}{2}(min(u^k, 0))^2$ in (67) and the function $\psi = min(u^k, 0)$ in (68). We see that $min(u^k, 0) \leq 0$ and moreover, $min(u^k, 0) = 0$ when $p_g^k \geq 0$. Therefore the integration in (67) and (68) is performed only in region where $p_g^k \leq 0$, and there, on the other hand, we have $\hat{\rho}_g^k = 0$. We denote $X = min(u^k, 0)$. By subtracting the liquid phase equation from the gas phase equation we get

$$\frac{1}{\delta t}\int_\Omega \Phi\left(X^2 S_l^k - (u^{k-1}S_l^{k-1} + \rho_g^{k-1}(1-S_l^{k-1}))X - (S^k - S^{k-1})\frac{X^2}{2}\right) dx$$
$$+\int_\Omega \Phi S_l^k D|\nabla X|^2\, dx + \int_\Omega S_l^k F_P^k \frac{X^2}{2}\, dx = -\int_\Omega F_I^k \frac{X^2}{2}\, dx.$$

It is easy to see that $X = 0$ which proves Lemma 6.

Now the existence of a unique solution of the problem (65)-(66) follows. This completes the proof of Theorem 2.

5 Proof of Theorem 1

The goal of this Section is to prove the main result of the paper, Theorem 1. We will achieve this by establishing estimates, uniform in δt, of solutions to the discrete system (65)-(66), and then passing to the limit as $\delta t \to 0$ in this system.

5.1 Uniform estimates with respect to δt

Note that the estimate (84) implies that each solution of (70), (71) satisfies

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\mathscr{E}^\varepsilon(p^\varepsilon, p_g^\varepsilon) - \mathscr{E}^\varepsilon(p^*, p_g^*)\right) dx + \int_\Omega \left(|\nabla p^\varepsilon|^2 + \frac{\hat{\rho}_g^\varepsilon}{\lambda}\mathbb{K}\nabla\theta^\varepsilon\cdot\nabla\theta^\varepsilon + |\nabla p_g^\varepsilon|^2 + |\nabla u^\varepsilon|^2\right) dx \leq C,$$

where a constant C is independent of δt and ε. By using weak lower semicontinuity of norms, at the limit as $\varepsilon \to 0$ we further obtain for the same constant C (independent of δt):

$$\frac{1}{\delta t}\int_\Omega \Phi\left(\mathscr{E}(p, p_g) - \mathscr{E}(p^*, p_g^*)\right) dx + \int_\Omega \left(|\nabla p|^2 + |\sqrt{\frac{\hat{\rho}_g(p_g)}{\lambda}}\nabla\theta|^2 + |\nabla p_g|^2 + |\nabla u|^2\right) dx \leq C.$$

Since this estimate holds for all time levels k, we write it for any fixed k, multiply by δt and sum over all time steps. This gives us that there is a constant C, independent of δt, such that the solutions $(p^{\delta t}, p_g^{\delta t})$ of (65)-(66) satisfy

$$\int_\Omega \left\{|\sqrt{\frac{\hat{\rho}_g(p_g)}{\lambda}}\nabla\theta^{\delta t}|^2 + |\nabla p^{\delta t}|^2 + |\nabla p_g^{\delta t}|^2 + |\nabla u^{\delta t}|^2\right\} dx \leq C. \tag{93}$$

For fixed k denote $r_g^k = \hat{u}(p_g^k)S_l^k + \hat{\rho}(p_g^k)(1-S_l^k)$, and corresponding piecewise constant and piecewise linear in time functions by $r_g^{\delta t}$ and $\tilde{r}_g^{\delta t}$. From (93) we conclude that $r_g^{\delta t}$ and $\tilde{r}_g^{\delta t}$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$.

5.2 Passage to the limit as $\delta t \to 0$

Uniform in δt estimates from the previous Subsection imply the following convergencies:

Lemma 7 *Let* $(p^{\delta t}, p_g^{\delta t})$ *be the solution of* (65)-(66). *There there exist functions* $p, p_g \in L^2(\Omega)$ *such that on a subsequence as* $\varepsilon \to 0$ *it holds:*

$$p^{\delta t} \to p \text{ weakly in } L^2(0,T;V); \tag{94}$$

$$p_g^{\delta t} \to p_g \text{ weakly in } L^2(0,T;V) \text{ and a. e. in } Q_T; \tag{95}$$

$$S^{\delta t} \to S \text{ strongly in } L^2(Q_T) \text{ and a. e. in } Q_T; \tag{96}$$

$$u^{\delta t} \to u := \hat{u}(p_g) \text{ weakly in } L^2(0,T;V); \tag{97}$$

$$r_g^{\delta t} \to \hat{u}(p_g)S + \hat{\rho}_g(p_g)(1-S) \text{ strongly in } L^2(Q_T) \text{ and a. e. in } Q_T. \tag{98}$$

Moreover, $0 \leq S \leq 1$ *and*

$$\Phi \partial_t \tilde{S}^{\delta t} \to \phi \partial_t S \text{ weakly in } L^2(0,T;H^{-1}(\Omega)); \tag{99}$$

$$\Phi \partial_t \tilde{r_g}^{\delta t} \to \phi \partial_t (\hat{u}(p_g)S + \hat{\rho}_g(p_g)(1-S)) \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{100}$$

Lemma 7 with boundedness of nonlinear coefficients allows us to pass to the limit as δt tends to 0 in system (65)-(66). This completes proof of Theorem 1.

6 Acknowledgement

This research has been supported by the Croatian Science Foundation, project number UIP-2017-05-7249 (MANDphy). Their support is gratefully acknowledged.

References

[1] B. Amaziane, M. Jurak, L. Pankratov, A. Piatnitski: An existence result for nonisothermal immiscible incompressible two-phase flow in heterogeneous porous media. Math. Meth. Appl. Sci. Vol. 40, No. 18, 7510-7539 (2017)

[2] B. Amaziane, M. Jurak: A new formulation of immiscible compressible two-phase flow in porous media, C. R. Mecanique 336 (2008) 600-605.

[3] B. Amaziane, M. Jurak, A. Vrbaški: Existence for a global pressure formulation of water-gas flow in porous media. Electron. J. Diff. Equ., Vol. 2012 (2012), No. 102, pp. 1-22

[4] B. Amaziane, M. Jurak, A. Vrbaški: Homogenization results for a coupled system modeling immiscible compressible two-phase flow in porous media by the concept of global pressure, Applicable Analysis Vol. 92, 7 (2013), pp. 1417-1433

[5] B. Amaziane, M. Jurak, A. Žgaljić Keko: Modeling and numerical simulations of immiscible compressible two-phase flow in porous media by the concept of global pressure, Transport in Porous Media (2009) DOI: 10.1007/s11242-009-9489-8.

[6] B. Amaziane, M. Jurak, A. Žgaljić Keko: An existence result for a coupled system modeling a fully equivalent global pressure formulation for immiscible compressible two-phase flow in porous media, J. Differential Equation 250, No 3 (2011), pp. 1685-1718.

[7] B. Amaziane, M. Jurak, A. Žgaljić Keko: Modeling compositional compressible two-phase flow in porous media by the concept of the global pressure, Comput Geosci (2014) 18: 297-309

[8] O. Angelini, C. Chavant, E. Chénier, R. Eymard, S. Granet: Finite Volume Approximation of a Diffusion-dissolution model and application to nuclear waste storage, Mathematics & Computers in simulation, 81, 2001-2017 (2011).

[9] J. Bear: Dynamics of Fluids in Porous Media. American Elsevier (1979).

[10] J. Bear, Y. Bachmat: Introduction to Modeling of Transport Phenomena in Porous Media. Kluwer, Dordrecht (1991).

[11] A. Bourgeat, M. Jurak, F. Smaï: Two partially miscible flow and transport modeling in porous media; application to gas migration in a nuclear waste repository. Comput. Geosci. 13(1), 29-42 (2009).

[12] A. Bourgeat, M. Jurak, F. Smaï: On persistent primary variables for numerical modeling of gas migration in a nuclear waste repository, Comput Geosci Vol. 17, 2, 287-305 (2013).

[13] R. J. Brooks, A. T. Corey: Properties of porous media affecting fluid flow. *J. Irrigation Drainage Div.*, in: Proc. Am Soc Civil Eng (IR2) 92, 61-88 (1966).

[14] F. Caro, B. Saad, M. Saad: Two-Component Two-Compressible Flow in a Porous Medium. Acta Appl Math 117, 15-46 (2012).

[15] F. Caro, B. Saad, M. Saad: Study of degenerate parabolic system modeling the hydrogen displacement in a nuclear waste repository. DCDS S, Vol. 7, 2, 191-205 (2014).

[16] G. Chavent, J. Jaffré: Mathematical Models and Finite Elements for Reservoir Simulation. North-Holland (1986).

[17] L. Esposito, C. Nitsch, C. Trombetti: Best Constants in Poincaré Inequalities for Convex Domains. https://arxiv.org/abs/1110.2960

[18] M. Van Genuchten: A closed form equation for predicting the hydraulic conductivity of unsaturated soils. Soil Sci. Soc. Am. J. 44, 892-898 (1980).

[19] M. Jurak, I. Radišić, A. Žgaljić Keko: Two-phase Two-component Flow in Porous Media in Low Solubility Regime SIAM J. Math. Anal., 51(3), 2019–2052.

[20] Z. Khalil, M. Saad: Solutions to a model for compressible immiscible two phase flow in porous media. Electronic Journal of Differential Equations Vol. 2010, 1-33 (2010)

[21] Z. Khalil, M. Saad: On a fully degenerate parabolic system modeling immiscible gas-water displacement in porous media. Nonlinear Analysis: Real World Applications 12, 1591-1615 (2011)

[22] C. M. Marle: Multiphase Flow in Porous Media. Édition Technip (1981).

[23] Y. Mualem: A New Model for Predicting the Hydraulic Conductivity of Unsaturated Porous Media. Water Resour. Res. 12(3), 513-522 (1976).

[24] Yu-Shu Wua, Forsyth, P.A.: On the selection of primary variables in numerical formulation for modeling multiphase flow in porous media. J. Cont. Hydr. 48, 277-304 (2001).