Discretization schemes for the two simplified global double porosity models of immiscible incompressible two-phase flow

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Abstract

We present the discretization schemes for the two simplified homogenized models of immiscible incompressible two-phase flow in double porosity media with thin fractures. The two models were derived previously by the authors by different linearizations of the nonlinear local problem called the imbibition equation which appears in the homogenized model after passage to the limit as $\varepsilon \to 0$. The models are fully homogenized with the matrix-fracture source terms expressed as a convolution.

1 Introduction

A naturally fractured reservoir is a porous medium consisting of two superimposed continua, a discontinuous system of periodically repeating medium-sized matrix blocks interlaced on a fine scale by a connected system of thin fissures. The fractures are notably more permeable than the porous matrix blocks. Hence, the reservoir's effective permeability is increased with respect to the permeability of the merely rock matrix. The transport of fluids through the reservoir primarily takes place within the fracture system where the flow is much readier than in the porous rock. On the other hand, the matrix stores most of the fluid. Moreover, the fractures are typically very narrow compared to the size of the reservoir. These contrasts cause great difficulties in numerical modelling of multiphase flow in such media because neither fractures nor matrix, or their interactions must not be neglected in a model of the flow. This type of porous medium is frequently encountered in hydrology and petroleum applications, for example improved oil recovery in hydrocarbon reservoirs, subsurface remediation or the disposal of radioactive waste.

Dual-continuum or double porosity models of multiphase flow in fractured porous media do not consider individual fractures but a network of small interconnected fractures. Two two large differences in scale are present in such systems: fracture width is very small compared to the scale of the domain and fracture permeability is much larger than the permeability of surrounding material. Certain averaging or homogenization processes are applied in order to obtain simplified description of matrix-fracture system and its interactions on a global scale of a reservoir.

The method of two small parameters ε, δ for homogenization of the two-phase flow in double porosity media was proposed in [**?**, **?**]. The parameter ε describes the periodicity of the fractured porous medium and δ is the relative width of the fractures. The homogenization process here consists of two steps: first homogenization as $\varepsilon \to 0$ and then one needs to pass to the limit as $\delta \to 0$. As further references on this procedure we refer to [1, 2] and the references therein. The first result on the homogenization of the two-phase flow in double porosity media was obtained in

[8] by the method of two small parameters. The method of two small parameters was later used also in [11] where a homogenized model of the two-phase non-equilibrium flow in fractured porous media was derived.

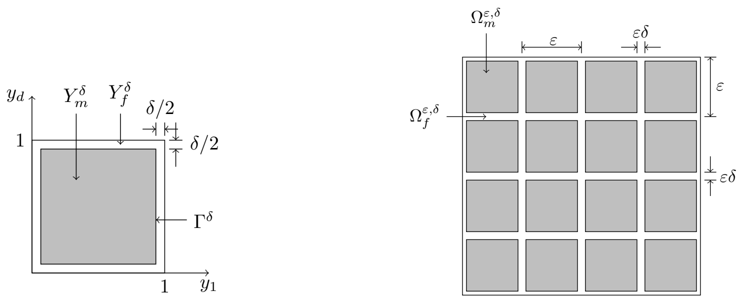
In the global double porosity δ-problem obtained in [8] after passing to the limit as $\varepsilon \to 0$ there appear additional matrix-fracture source terms that are defined implicitly via solutions of a nonlinear local boundary value problem known as the imbibition equation. The nonlinearity of the imbibition equation presents difficulties in numerical simulations of the model since no analytic expression of the matrix-fracture source term is available. In order to overcome this issue, one can linearize the imbibition equation and then express the matrix-fracture source term explicitly from the linearized equation. In [8] the imbibition equation is linearized by using an appropriate constant, as suggested by [3]. Then in [9] we present a new, variable and more general linearization of the imbibition equation which gives a new simplified double porosity model. In this paper we provide the dicretization schemes for the two simplified double porosity models obtained by the two linearizations.

The rest of the paper is organized as follows. In Section 2 we present the two fully homogenized simplified double porosity models of the two-phase flow in the case of thin fractures. The main contributions of the paper are contained in Sections 3 and 4 in which we propose the discretization schemes of the effective system including the matrix-fracture source terms in the convolution form obtained by constant linearization and by variable linearization, respectively.

2 Global simplified double porosity models I and II

In this section we present two global double porosity - type models of incompressible two-phase flow in fractured porous media derived previously by the authors in [8] and [9], respectively. In Sections 3 and 4 we will present the numerical schemes for these models.

Let the reservoir $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, and we denote $\Omega_T = \Omega \times (0, T)$, where $T > 0$ is fixed. In the derivation of our models it is assumed that the reservoir Ω is composed of the matrix blocks and the highly permeable connected network of fractures. The characteristic matrix block size ε is small compared to the size of the flow domain, i.e., $\varepsilon \ll 1$ is a small parameter describing the periodicity of the fractured porous medium. The thickness of the fractures is supposed to be of order $\varepsilon\delta$, where $0 < \varepsilon \ll \delta < 1$ is a second small parameter, describing the relative thickness or opening of the fractures. The reference cell is $Y = (0, 1)^d$ and the domain Ω is taken to be covered by a pavement of cells εY (see Figure 1). The porosities of the blocks and the fractures are supposed to be constant and are denoted by Φ_m and Φ_f, respectively. The permeabilities of the blocks and the fractures are highly contrasted. In the double porosity model it is assumed that the fracture porosity is k_f, while the matrix porosity is $(\varepsilon\delta)^2 k_m$, where k_f and k_m are of the same order.



(a) Reference cell: $Y =]0, 1[^d$ (b) Domain: $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

Figure 1: The geometry of the reservoir Ω

The starting point in derivation of our global simplified models is the system describing the two-phase incompressible flow in Ω_T, which depends on two small parameters ε and δ. In order to obtain the effective model, the first step is to pass to the limit as $\varepsilon \to 0$, which has been proved rigorously in [4, 5, 12]. The resulting global δ - double porosity model is not fully homogenized (in the sense of [10]) since the effective permeability and the matrix-fracture source term are given through some unknown functions which are defined as the solutions of the coupled local problems. However, by letting the small parameter δ to zero one can obtain full decoupling of the local and the global problem. In the case of the two-phase flow that problem was first studied in [8] by performing the linearization of the imbibition equation by a constant, as suggested by [3] (see the details in [8]). It is shown in [8] that in the limit as $\delta \to 0$ the effective fracture equations coupled with the linearized imbibition equation obtained by this constant linearization reduce to the system:

$$\begin{cases} \Phi_f \dfrac{\partial S}{\partial t} - \operatorname{div}\Big(k^* \lambda_{w,f}(S) \nabla P_w\Big) = \mathcal{Q}_w \\ -\Phi_f \dfrac{\partial S}{\partial t} - \operatorname{div}\Big(k^* \lambda_{n,f}(S) \nabla P_n\Big) = \mathcal{Q}_n, \\ P_{c,f}(S) = P_n - P_w. \end{cases} \tag{1}$$

Here S, P_w and P_n are the wetting phase saturation and pressure and non wetting phase pressure in the system of fractures, respectively; $P_{c,f}$, $\lambda_{w,f}$ and $\lambda_{n,f}$ are the capillary pressure function and the phase mobility functions in the fractures; $k^* = k_f(d-1)/d$ is the reduced fracture permeability. The terms $\mathcal{Q}_w$ and $\mathcal{Q}_n$ are the wetting phase and the non wetting phase source terms modeling the phase mass transfer from the matrix to the fracture system governed by the capillary imbibition and are given by

$$\mathcal{Q}_w(x,t) = -C_m \frac{\partial}{\partial t} \int_0^t \frac{\mathcal{P}(S(x,u)) - \mathcal{P}(S(x,0))}{\sqrt{t-u}}\, du = -\mathcal{Q}_n(x,t), \tag{2}$$

where $\mathcal{P}(S) \stackrel{\text{def}}{=} P_{c,m}^{-1}(P_{c,f}(S))$, $C_m = 2d\sqrt{\Phi_m k_m \overline{\alpha}_m/\pi}$, $\overline{\alpha}_m = \int_0^1 \alpha_m(s)\, ds$, and

$$\alpha_m(s) \stackrel{\text{def}}{=} \frac{\lambda_{w,m}(s)\, \lambda_{n,m}(s)}{\lambda_m(s)} |P'_{c,m}(s)|. \tag{3}$$

Here $P_{c,m}$, $\lambda_{w,m}$ and $\lambda_{n,m}$ are the capillary pressure function and the phase mobility functions in the matrix and the total mobility in the matrix $\lambda_m(s) = \lambda_{w,m}(s) + \lambda_{n,m}(s)$.

Let us note that in this final effective model the effective porosity is the fracture porosity, the effective permeability is reduced fracture permeability, and the matrix-fracture source term $\mathcal{Q}_w(t)$ is expressed as the convolution (2) with the kernel $\mathcal{K}(t) = C_m/\sqrt{t}$. The model (1), (2) (*double porosity model I*) is fully homogenized, in other words the local and the global problems are completely decoupled, since the effective permeability and the matrix-fracture source term are given explicitly and there is no need to solve local problems.

The new, more general approach to the linearization of the imbibition equation presented in [9] leads to our second effective model (*double porosity model II*). In this case a variable linearization of the imbibition equation was proposed. The effective model consists of the system (1) with the effective matrix-fracture source term $\widetilde{\mathcal{Q}}_w = -\widetilde{\mathcal{Q}}_n$ given by

$$\widetilde{\mathcal{Q}}_w(x,t) = -\widetilde{C_m} \frac{\partial}{\partial t} \int_0^t \frac{\beta_m(\mathcal{P}(S(x,s))) - \beta_m(\mathcal{P}(S(x,0))}{\sqrt{\tau_x(t) - \tau_x(s)}}\, ds. \tag{4}$$

Here $\widetilde{C_m} = 2d\sqrt{\Phi_m k_m/\pi}$ and

$$\tau_x(t) \stackrel{\text{def}}{=} \int_0^t \frac{\beta_m(\mathcal{P}(S(x,s))) - \beta_m(\mathcal{P}(S(x,0)))}{\mathcal{P}(S(x,s)) - \mathcal{P}(S(x,0))}\, ds, \tag{5}$$

with

$$\beta_m(s) \stackrel{\text{def}}{=} \int_0^s \alpha_m(\xi)\, d\xi. \tag{6}$$

We note that linearization of the imbibition equation by a constant is a special case of this new linearization by using a variable coefficient.

Remark 1 *Numerical simulations (for the details see [9]) have shown that the effective matrix-fracture exchange source term* (4) *obtained by the new variable linearization in [9] gives a much better approximation of the exact one than the corresponding effective matrix-fracture exchange source term* (2) *obtained previously by a constant linearization in [8].*

3 Discretization of double porosity model I

The model (1)–(2) will be discretized by the cell centered finite volume method on a structured grid with the two-point flux approximation. First we present the time discretization.

Assume that we have a sequence of time steps: $0 = t^0 < t^1 < \cdots < t^n < \cdots$ and denote $\delta t^n = t^{n+1} - t^n$ and also $I_n = (t_{n-1}, t_n]$. All unknowns are supposed to be piecewise constant in time, such that $S(x,t) = \sum_k S^k(x)\chi_{I_k}(t)$, where $S^k(x) = S(x, t_k)$, and similarly for other variables. Implicit Euler discretization gives for $t \in I_{n+1}$,

$$\Phi_f \frac{S^{n+1} - S^n}{\delta t^n} - \operatorname{div}\left(\lambda_{w,f}(S^{n+1}) k^* \nabla P_w^{n+1}\right) = Q^{n+1/2},$$

$$-\Phi_f \frac{S^{n+1} - S^n}{\delta t^n} - \operatorname{div}\left(\lambda_{n,f}(S^{n+1}) k^* \nabla P_n^{n+1}\right) = -Q^{n+1/2}.$$

The source term is discretized in the following way:

$$\begin{aligned}
Q^{n+1/2} &= -\frac{C_m}{\delta t^n}\Big(\int_0^{t_{n+1}} \sum_{k=1}^{n+1} \frac{\mathcal{P}(S^k) - \mathcal{P}(S^0)}{\sqrt{t_{n+1} - s}} \chi_{I_k}(s)\, ds \\
&\qquad - \int_0^{t_n} \sum_{k=1}^{n} \frac{\mathcal{P}(S^k) - \mathcal{P}(S^0)}{\sqrt{t_n - s}} \chi_{I_k}(s)\, ds \Big) \\
&= -\frac{1}{\delta t^n}\Big(\sum_{k=1}^{n+1} (\mathcal{P}(S^k) - \mathcal{P}(S^0)) \int_{t_{k-1}}^{t_k} \frac{C_m ds}{\sqrt{t_{n+1} - s}} \\
&\qquad - \sum_{k=1}^{n} (\mathcal{P}(S^k) - \mathcal{P}(S^0)) \int_{t_{k-1}}^{t_k} \frac{C_m ds}{\sqrt{t_n - s}} \Big) \\
&= -\frac{C_m}{\delta t^n}\Big(\sum_{k=1}^{n+1} (\mathcal{P}(S^k) - \mathcal{P}(S^0)) I_k^{n+1} - \sum_{k=1}^{n} (\mathcal{P}(S^k) - \mathcal{P}(S^0)) I_k^n \Big) \\
&= -\frac{C_m}{\delta t^n}\Big((\mathcal{P}(S^{n+1}) - \mathcal{P}(S^0)) I_{n+1}^{n+1} + \sum_{k=1}^{n} (\mathcal{P}(S^k) - \mathcal{P}(S^0))(I_k^{n+1} - I_k^n) \Big)
\end{aligned}$$

where we denoted for $1 \le k \le n$,

$$I_k^n = \int_{t_{k-1}}^{t_k} \frac{C_m ds}{\sqrt{t_n - s}} = 2C_m(\sqrt{t_n - t_{k-1}} - \sqrt{t_n - t_k}) = \frac{\delta t^{k-1}}{\sqrt{t_n - t_{k-1}} + \sqrt{t_n - t_k}}.$$

Obviously, $I_{n+1}^{n+1} = 2C_m\sqrt{\delta t^n}$. If the time grid is equidistant, then we have $I_{k+1}^{n+1} = I_k^n = J_{n-k}$, since

$$\int_{t_k}^{t_{k+1}} \frac{ds}{\sqrt{t_{n+1} - s}} = \int_{t_{k-1}}^{t_k} \frac{ds}{\sqrt{t_n - s}} = \frac{2\sqrt{\delta t}}{\sqrt{n-k+1} + \sqrt{n-k}}, \quad J_l = \frac{2C_m\sqrt{\delta t}}{\sqrt{l+1} + \sqrt{l}},$$

leading to a convolution-like representation:

$$Q^{n+1/2} = -\frac{1}{\delta t} \sum_{k=0}^{n} [\mathcal{P}(S^{k+1}) - \mathcal{P}(S^k)] J_{n-k}.$$

Generally we have:

$$\Phi_f \frac{S^{n+1}}{\delta t^n} + \frac{2C_m}{\sqrt{\delta t^n}} \mathcal{P}(S^{n+1}) - \text{div} \left(\lambda_{w,f}(S^{n+1}) k^* \nabla P_w^{n+1} \right) = \Phi_f \frac{S^n}{\delta t^n} + \frac{1}{\delta t^n} \mathcal{F}^n \tag{7}$$

$$-\Phi_f \frac{S^{n+1}}{\delta t^n} - \frac{2C_m}{\sqrt{\delta t^n}} \mathcal{P}(S^{n+1}) - \text{div} \left(\lambda_{n,f}(S^{n+1}) k^* \nabla P_n^{n+1} \right) = -\Phi_f \frac{S^n}{\delta t^n} - \frac{1}{\delta t^n} \mathcal{F}^n \tag{8}$$

where, for $n > 0$,

$$\mathcal{F}^n = \mathcal{P}(S^0) I_{n+1}^{n+1} - \sum_{k=1}^{n} (\mathcal{P}(S^k) - \mathcal{P}(S^0))(I_k^{n+1} - I_k^n) \tag{9}$$

$$= \mathcal{P}(S^0)(I_{n+1}^{n+1} + \sum_{k=1}^{n} (I_k^{n+1} - I_k^n)) - \sum_{k=1}^{n} \mathcal{P}(S^k)(I_k^{n+1} - I_k^n). \tag{10}$$

Also, note that in the case $n = 0$ we have,

$$Q_w^{1/2} = -\frac{C_m}{\delta t^0} \Big(\int_0^{t_1} \frac{\mathcal{P}(S^1) - \mathcal{P}(S^0)}{\sqrt{t_1 - s}} \, ds - 0 \Big) = -\frac{C_m}{\delta t^0} (\mathcal{P}(S^1) - \mathcal{P}(S^0)) \int_0^{t_1} \frac{ds}{\sqrt{t_1 - s}}.$$

Therefore, for $n = 0$ we have

$$\mathcal{F}^0 = \mathcal{P}(S^0) I_1^1 = 2\sqrt{\delta t^0} \mathcal{P}(S^0),$$

and (9) holds also for $n = 0$.

Let us denote $D_k^n = I_k^n - I_k^{n+1}$ for $k = 1, \dots, n$. We have

$$D_k^n = I_k^n - I_k^{n+1} = \frac{C_m \delta t^{k-1}}{\sqrt{t_n - t_{k-1}} + \sqrt{t_n - t_k}} - \frac{C_m \delta t^{k-1}}{\sqrt{t_{n+1} - t_{k-1}} + \sqrt{t_{n+1} - t_k}} > 0$$

since the function

$$\omega(t) = \frac{C_m}{\sqrt{t}}$$

is monotone decreasing. We also introduce

$$D_0^n = I_{n+1}^{n+1} + \sum_{k=1}^{n} (I_k^{n+1} - I_k^n) = I_1^{n+1} + \sum_{k=1}^{n} (I_{k+1}^{n+1} - I_k^n).$$

Let us note that in the equidistant time stepping we have $I_{k+1}^{n+1} - I_k^n = 0$ and then $D_0^n = I_1^{n+1} > 0$. In the non equidistant case the terms $I_{k+1}^{n+1} - I_k^n$ can have any sign, so we will introduce the assumption that the time discretization is such that

$$D_0^n > 0. \tag{11}$$

This will always be the case if the time stepping is close to the equidistant one.

With introduced notation we can write

$$\mathcal{F}^n = \sum_{k=0}^{n} D_k^n \mathcal{P}(S^k) \tag{12}$$

with $D_k^n > 0$ for $k = 0, 1, \dots, n$.

Let us also note that

$$\sum_{k=0}^{n} D_k^n = \sum_{k=1}^{n}(I_k^n - I_k^{n+1}) + I_{n+1}^{n+1} + \sum_{k=1}^{n}(I_k^{n+1} - I_k^n) = I_{n+1}^{n+1} = 2C_m\sqrt{\delta t^n}. \tag{13}$$

We use standard finite volume discretization of the two phase system written in the phase formulation [7] (see [6] for notations):

$$\begin{aligned}\Phi_{f,K}\frac{S_K^{n+1}}{\delta t^n} + \frac{2C_m}{\sqrt{\delta t^n}}\mathcal{P}(S_K^{n+1}) - \sum_{L\in\mathcal{N}_K}\tau_{K|L}k^*_{K|L}\lambda^{n+1}_{w,f,K|L}\delta^{n+1}_{K,L}(P_w)\\ = \Phi_f\frac{S_K^n}{\delta t^n} + \frac{1}{\delta t^n}\mathcal{F}_K^n\end{aligned} \tag{14}$$

$$\begin{aligned}-\Phi_{f,K}\frac{S_K^{n+1}}{\delta t^n} - \frac{2C_m}{\sqrt{\delta t^n}}\mathcal{P}(S_K^{n+1}) - \sum_{L\in\mathcal{N}_K}\tau_{K|L}k^*_{K|L}\lambda^{n+1}_{n,f,K|L}\delta^{n+1}_{K,L}(P_n)\\ = -\Phi_{f,K}\frac{S_K^n}{\delta t^n} - \frac{1}{\delta t^n}\mathcal{F}_K^n\end{aligned} \tag{15}$$

In this discretization we use phase by phase upstream choice: the value of the mobility of each phase on the edge $K|L$ is determined by the sign of the difference of the discrete phase pressure.

$$\lambda^{n+1}_{w,f,K|L} = \lambda_{w,f}(S^{n+1}_{w,K|L}), \quad \lambda^{n+1}_{n,f,K|L} = \lambda_{n,f}(S^{n+1}_{n,K|L}), \tag{16}$$

with

$$\begin{aligned}&S^{n+1}_{w,K|L} = \begin{cases} S_K^{n+1} & \text{if } (K,L)\in\mathcal{E}_w^{n+1}\\ S_L^{n+1} & \text{otherwise,}\end{cases} \qquad S^{n+1}_{n,K|L} = \begin{cases} S_K^{n+1} & \text{if } (K,L)\in\mathcal{E}_n^{n+1}\\ S_L^{n+1} & \text{otherwise,}\end{cases}\\ &\text{where } \mathcal{E}_w^{n+1} \text{ and } \mathcal{E}_n^{n+1} \text{ are two subsets of } \mathcal{E} \text{ such that}\\ &\{(K,L)\in\mathcal{E}\colon \delta^{n+1}_{K,L}(P_w) < 0\} \subset \mathcal{E}_w^{n+1} \subset \{(K,L)\in\mathcal{E}\colon \delta^{n+1}_{K,L}(P_w) \le 0\}\\ &\{(K,L)\in\mathcal{E}\colon \delta^{n+1}_{K,L}(P_n) < 0\} \subset \mathcal{E}_n^{n+1} \subset \{(K,L)\in\mathcal{E}\colon \delta^{n+1}_{K,L}(P_n) \le 0\}\end{aligned} \tag{17}$$

4 Discretization of double porosity model II

Second model differs from the first one only in the matrix-fracture exchange term which takes the form:

$$\widetilde{\mathcal{Q}}_w(x,t) = -C_m\frac{\partial}{\partial t}\int_0^{\tau_x(t)}\frac{\mathcal{P}(S(x,(\tau_x)^{-1}(u))) - \mathcal{P}(S(x,0))}{\sqrt{\tau_x(t)-u}}\,du, \tag{18}$$

where $C_m = 2\sqrt{\Phi_m k_m/\pi}$ and τ_x is given by

$$\tau_x(t) = \int_0^t \widehat{\alpha}_m(x,s)\,ds.$$

We have chosen expression for the matrix-fracture exchange term given by (18) but it is also possible to use other forms, for example (4).

We will discretize this model using the same approach as in the constant linearization model. Assume that we have a sequence of time instances $0 = t^0 < t^1 < \dots < t^n < \cdots$ and denote by $\tau_x^n =$

$\tau_x(t^n)$. If the saturation S is constant in time on each interval (t^k, t^{k+1}) then $\mathcal{P}(S(x,(\tau_x)^{-1}(u))) - \mathcal{P}(S(x,0))$ is constant on each interval (τ_x^k, τ_x^{k+1}) and we can write

$$\begin{aligned}
\widetilde{\mathrm{Q}}_w^{n+1/2} &\approx -\frac{C_m}{\delta t^n}\Big(\int_0^{\tau_x^{n+1}} \frac{\mathcal{P}(S(x,(\tau_x)^{-1}(u))) - \mathcal{P}(S(x,0))}{\sqrt{\tau_x^{n+1}-u}}\,du \\
&\quad - \int_0^{\tau_x^n} \frac{\mathcal{P}(S(x,(\tau_x)^{-1}(u))) - \mathcal{P}(S(x,0))}{\sqrt{\tau_x^n-u}}\,du\Big) \\
&= -\frac{1}{\delta t^n}\Big(\sum_{k=1}^{n+1}\int_{\tau_x^{k-1}}^{\tau_x^k} \frac{C_m du}{\sqrt{\tau_x^{n+1}-u}}(\mathcal{P}(S^k(x)) - \mathcal{P}(S^0(x))) \\
&\quad - \sum_{k=1}^{n}\int_{\tau_x^{k-1}}^{\tau_x^k} \frac{C_m du}{\sqrt{\tau_x^{n}-u}}(\mathcal{P}(S^k(x)) - \mathcal{P}(S^0(x)))\Big).
\end{aligned}$$

As before we have

$$I_k^n = \int_{\tau_x^{k-1}}^{\tau_x^k} \frac{C_m du}{\sqrt{\tau_x^n - u}} = 2C_m\left(\sqrt{\tau_x^n - \tau_x^{k-1}} - \sqrt{\tau_x^n - \tau_x^k}\right) = \frac{2C_m(\tau_x^k - \tau_x^{k-1})}{\sqrt{\tau_x^n - \tau_x^{k-1}} + \sqrt{\tau_x^n - \tau_x^k}}.$$

Note that for $s \in (t^{k-1}, t^k)$ we have

$$\tau(s) = \int_0^s \widehat{\alpha}_m(u)\,du = \sum_{l=1}^{k-1} \widehat{\alpha}_m^l \delta t^{l-1} + \widehat{\alpha}_m^k (s - t^{k-1}),$$

where $\widehat{\alpha}_m^l = \widehat{\alpha}_m(t^l)$, so that

$$\tau(t^n) - \tau(t^{k-1}) = \sum_{l=k}^{n} \widehat{\alpha}_m^l \delta t^{l-1}, \quad \tau(t^n) - \tau(t^k) = \sum_{l=k+1}^{n} \widehat{\alpha}_m^l \delta t^{l-1}, \quad \tau(t^{k+1}) - \tau(t^k) = \widehat{\alpha}_m^k \delta t^{k-1}.$$

Therefore, we have for $k \le n$,

$$I_k^n = \frac{2C_m \widehat{\alpha}_m^k \delta t^{k-1}}{\sqrt{\sum_{l=k}^n \widehat{\alpha}_m^l \delta t^{l-1}} + \sqrt{\sum_{l=k+1}^n \widehat{\alpha}_m^l \delta t^{l-1}}}.$$

For notational simplicity we will introduce for $k \le n$

$$U_k^n = \sum_{l=k}^{n} \widehat{\alpha}_m^l \delta t^{l-1}, \tag{19}$$

and $U_k^n = 0$ for $k > n$. Then we can write:

$$\widetilde{\mathrm{Q}}_w^{n+1/2} = -\frac{2C_m}{\delta t^n}\Big(\sum_{k=1}^{n+1} \frac{\widehat{\alpha}_m^k(\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^{n+1}} + \sqrt{U_{k+1}^{n+1}}}\delta t^{k-1} - \sum_{k=1}^{n} \frac{\widehat{\alpha}_m^k(\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^n} + \sqrt{U_{k+1}^n}}\delta t^{k-1}\Big).$$

We finally obtain the following scheme:

$$\begin{aligned}
\Phi_f \frac{S^{n+1}}{\delta t^n} &+ \frac{2C_m}{\delta t^n}\sum_{k=1}^{n+1} \frac{\widehat{\alpha}_m^k(\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^{n+1}} + \sqrt{U_{k+1}^{n+1}}}\delta t^{k-1} - \operatorname{div}\left(\lambda_{w,f}(S^{n+1}) k^* \nabla P_w^{n+1}\right) \\
&= \Phi_f \frac{S^n}{\delta t^n} + \frac{2C_m}{\delta t^n}\sum_{k=1}^{n} \frac{\widehat{\alpha}_m^k(\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^n} + \sqrt{U_{k+1}^n}}\delta t^{k-1}
\end{aligned} \tag{20}$$

$$
\begin{aligned}
&-\Phi_f \frac{S^{n+1}}{\delta t^n} - \frac{2C_m}{\delta t^n} \sum_{k=1}^{n+1} \frac{\widehat{\alpha}_m^k (\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^{n+1}} + \sqrt{U_{k+1}^{n+1}}} \delta t^{k-1} - \operatorname{div}\left(\lambda_{n,f}(S^{n+1}) k^* \nabla P_n^{n+1} \right) \\
&\qquad = -\Phi_f \frac{S^n}{\delta t^n} - \frac{2C_m}{\delta t^n} \sum_{k=1}^{n} \frac{\widehat{\alpha}_m^k (\mathcal{P}(S^k) - \mathcal{P}(S^0))}{\sqrt{U_k^n} + \sqrt{U_{k+1}^n}} \delta t^{k-1}
\end{aligned}
\tag{21}
$$

Note that

$$
\widehat{\alpha}_m^k = \frac{\beta_m(\mathcal{P}(S_{\max}^k)) - \beta_m(\mathcal{P}(S_{\min}^k))}{\mathcal{P}(S_{\max}^k) - \mathcal{P}(S_{\min}^k)}.
$$

where

$$
S_{\max}^k(x) = \max_{0 \le j \le k} S^j(x), \quad S_{\min}^k(x) = \min_{0 \le j \le k} S^j(x).
$$

Using standard finite volume discretization of the two phase system written in the phase formulation (see [7]) we get

$$
\begin{aligned}
&\Phi_{f,K} \frac{S_K^{n+1}}{\delta t^n} + \frac{2C_m}{\delta t^n} \sum_{k=1}^{n+1} \frac{\widehat{\alpha}_{m,K}^k (\mathcal{P}(S_K^k) - \mathcal{P}(S_K^0))}{\sqrt{U_{k,K}^{n+1}} + \sqrt{U_{k+1,K}^{n+1}}} \delta t^{k-1} - \sum_{L \in \mathcal{N}_K} \tau_{K|L} k_{K|L}^* \lambda_{w,f,K|L}^{n+1} \delta_{K,L}^{n+1}(P_w) \\
&\qquad = \Phi_f \frac{S_K^n}{\delta t^n} + \frac{2C_m}{\delta t^n} \sum_{k=1}^{n} \frac{\widehat{\alpha}_{m,K}^k (\mathcal{P}(S_K^k) - \mathcal{P}(S_K^0))}{\sqrt{U_{k,K}^n} + \sqrt{U_{k+1,K}^n}} \delta t^{k-1},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&-\Phi_{f,K} \frac{S_K^{n+1}}{\delta t^n} - \frac{2C_m}{\delta t^n} \sum_{k=1}^{n+1} \frac{\widehat{\alpha}_{m,K}^k (\mathcal{P}(S_K^k) - \mathcal{P}(S_K^0))}{\sqrt{U_{k,K}^{n+1}} + \sqrt{U_{k+1,K}^{n+1}}} \delta t^{k-1} - \sum_{L \in \mathcal{N}_K} \tau_{K|L} k_{K|L}^* \lambda_{n,f,K|L}^{n+1} \delta_{K,L}^{n+1}(P_n) \\
&\qquad = -\Phi_{f,K} \frac{S_K^n}{\delta t^n} - \frac{2C_m}{\delta t^n} \sum_{k=1}^{n} \frac{\widehat{\alpha}_{m,K}^k (\mathcal{P}(S_K^k) - \mathcal{P}(S_K^0))}{\sqrt{U_{k,K}^n} + \sqrt{U_{k+1}^n}} \delta t^{k-1,K},
\end{aligned}
\tag{23}
$$

where

$$
U_{k,K}^n = \sum_{l=k}^{n} \widehat{\alpha}_{m,K}^l \delta t^{l-1}, \tag{24}
$$

and

$$
\widehat{\alpha}_{m,K}^k = \frac{\beta_m(\mathcal{P}(S_{\mathrm{max,K}}^k)) - \beta_m(\mathcal{P}(S_{\mathrm{min,K}}^k))}{\mathcal{P}(S_{\mathrm{max,K}}^k) - \mathcal{P}(S_{\mathrm{min,K}}^k)}. \tag{25}
$$

where

$$
S_{\mathrm{max,K}}^k = \max_{0 \le j \le k} S_K^j, \quad S_{\mathrm{min,K}}^k = \min_{0 \le j \le k} S_K^j.
$$

5 Conclusion

We study the double porosity models (1), (2) and (1), (4) of the incompressible two-phase flow in fractured porous media which are obtained after a standard periodic homogenization as $\varepsilon \to 0$ and then linearizing the nonlinear imbibition equation in a two different ways: by replacing the nonlinear function (3) by a constant (as in [8], based on the idea of [3]), and by replacing (3) by a variable function, as in [9]. The need for the linearization of the imbibition equation comes from the fact that its nonlinearity causes difficulties for numerical simulations since the source terms at each point of the domain Ω are given through solutions of the imbibition equation so the associated boundary value problem needs to be solved many times. In models (1), (2) and (1), (4) the coefficients of the system are given explicitly and for their calculation solving local problems is no more required. In this work we present discretization schemes by the cell centered finite volume method for the effective system in both cases of linearization. Special care is taken of the matrix-fracture source terms which are given in a form of a convolution. The numerical simulations of the simplified double porosity model are a part of our future research.

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