Mathematical analysis of Barenblatt's model of non-equilibrium two-phase flow in porous media

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Abstract

We consider Barenblatt's model of non-equilibrium immiscible two-phase incompressible flow in porous media. In this model it is assumed that the relative phase permeabilities and the capillary pressure functions depend on the effective saturation η rather than on the actual saturation S. Using the concept of global pressure we derive the existence of weak solutions for the regularized two-phase incompressible Barenblatt's non-equilibrium flow problem.

1 Introduction

Classical mathematical models of two-phase fluid flow through porous media are based on the fundamental assumption of local phase equilibrium. In this model, the two phase flows are locally redistributed over their flow paths similarly to steady flows. Under this assumption, the relative phase permeabilities and the capillary pressure are taken to depend solely on the local water saturation, which leads to a closed system of equations for fluid phase velocities and pressures, and water saturation. This model has been widespread used for numerical predictions of water-oil displacement. However, in some of the crucial mechanisms of secondary oil recovery, such as forced oil-water displacement and spontaneous countercurrent imbibition, the characteristic times of these processes are, in general, comparable with the times of redistribution of flow paths between oil and water and therefore the non-equilibrium effects should be taken into account. Kondaurov proposed in [13] an approach to modelling non-equilibrium effects which is based on the non-equilibrium thermodynamics principles. This model has been further investigated in [14], [15], [16], [17]. Another model has been introduced by Barenblatt in [3] and further investigated in a number of papers, see for instance [3], [6], [5]. Here it is assumed that the relative permeability and the capillary pressure functions depend not on the actual instantaneous saturation S, but on an effective saturation η that corresponds to the equilibrium value that might be reached by the system after a phase redistribution in space. This makes the mathematical analysis of the problem more involved. Up to our knowledge, there is no rigorous mathematical studies of the Barenblatt's model. In this work, we derive the result on the existence of weak solutions for the regularized two-phase incompressible Barenblatt's non-equilibrium flow problem.

2 Mathematical model

2.1 The description of the model

We consider a reservoir $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) which is assumed to be a bounded, connected domain with sufficiently smooth boundary. Before describing the equations of the model, we give the notation.

- $\Phi = \Phi(x)$ is the porosity of the reservoir Ω;
- $K = K(x)$ is the absolute permeability tensor of the reservoir Ω;
- $S = S_w = S_w(x,t)$, $S_n = S_n(x,t)$ are the saturations of wetting and nonwetting fluids in the reservoir Ω, respectively;
- $p_w = p_w(x,t)$, $p_n = p_n(x,t)$ are the pressures of wetting and nonwetting fluids in the reservoir Ω, respectively.

Following the concept of Barenblatt's non-equilibrium model (see [6]), we suppose that for a non-equilibrium process there exists a relaxation (redistribution) time τ, $0 < \tau \ll 1$, and an effective saturation $\boldsymbol{\eta}$, generally different from the actual saturation S, such that $f^n(S) = f^e(\boldsymbol{\eta})$, where f^n and f^e denote non-equilibrium and equilibrium functions, respectively. The effective saturation $\boldsymbol{\eta}$ corresponds to the equilibrium value that might be reached by the system after a phase redistribution in space. In this model the relative permeabilities and the capillary pressure functions depend on $\boldsymbol{\eta}$ rather than on S. The notation is as follows:

- $\boldsymbol{\eta} = S + \tau \frac{\partial S}{\partial t}$ is the Barenblatt effective saturation;
- $k_{r,w} = k_{r,w}(\boldsymbol{\eta})$, $k_{r,n} = k_{r,n}(\boldsymbol{\eta})$ are the relative permeabilities of wetting and nonwetting fluids in the reservoir Ω, respectively;
- $P_c = P_c(\boldsymbol{\eta})$ is the capillary pressure function;
- $\mathbf{W}_w = K(x)\lambda_w(\boldsymbol{\eta})\nabla p_w$, $\mathbf{W}_n = K(x)\lambda_n(\boldsymbol{\eta})\nabla p_n$ are the phase fluxes; $\mathbf{W} \overset{\text{def}}{=} \mathbf{W}_w + \mathbf{W}_n$.

For the sake of simplicity, we assume no source/sink term and neglect the gravity effects. Then the system describing the immiscible flow of two incompressible fluids in a porous medium is obtained from the conservation of mass with the Darcy-Muskat law in each phase and it can be written as:

$$
\begin{cases}
\Phi \dfrac{\partial S}{\partial t} - \operatorname{div}\big(K(x)\lambda_w(\boldsymbol{\eta})\nabla p_w\big) = 0 \quad \text{in } \Omega_T, \\
-\Phi \dfrac{\partial S}{\partial t} - \operatorname{div}\big(K(x)\lambda_n(\boldsymbol{\eta})\nabla p_n\big) = 0 \quad \text{in } \Omega_T, \\
P_c(\boldsymbol{\eta}) = p_n - p_w \quad \text{in } \Omega_T, \\
\boldsymbol{\eta} = S + \tau \dfrac{\partial S}{\partial t} \quad \text{in } \Omega_T,
\end{cases}
\tag{2.1}
$$

where $\Omega_T \overset{\text{def}}{=} \Omega \times (0, T)$ with a given $T > 0$ and where the phase mobilities are given by

$$
\lambda_w(\boldsymbol{\eta}) \overset{\text{def}}{=} \frac{k_{r,w}}{\mu_w}(\boldsymbol{\eta}); \quad \lambda_n(\boldsymbol{\eta}) \overset{\text{def}}{=} \frac{k_{r,n}}{\mu_n}(\boldsymbol{\eta}).
$$

We denote

$$\lambda(\boldsymbol{\eta}) \stackrel{\text{def}}{=} \lambda_w(\boldsymbol{\eta}) + \lambda_n(\boldsymbol{\eta}).$$

Here and below

$$S \stackrel{\text{def}}{=} S_w.$$

We complete the system (2.1) by the corresponding boundary and initial conditions.

Boundary conditions. We suppose that the boundary $\partial\Omega$ consists of two disjoint parts Γ_1 and Γ_2 such that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The boundary conditions are given by:

$$\begin{cases} p_w(x,t) = p_n(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,T); \\ \mathbf{W}_w \cdot \vec{\nu} = \mathbf{W}_n \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_2 \times (0,T). \end{cases} \tag{2.2}$$

Here $\vec{\nu}$ is the unit outer normal on Γ_2.

Initial conditions. The initial conditions read:

$$\begin{cases} S(x,0) = S^0(x) \quad \text{and} \quad \boldsymbol{\eta}(x,0) = \boldsymbol{\eta}^0(x) = S^0(x) \quad \text{in } \Omega; \\ p_w(x,0) = p_w^0(x), \quad p_n(x,0) = p_n^0(x) \quad \text{in } \Omega. \end{cases} \tag{2.3}$$

Remark 1 *In this work we consider the case of the initial equilibrium state, which corresponds to the initial condition for* $\boldsymbol{\eta}$ *in* (2.3).

Remark 2 *Let us notice that in the case of the equilibrium flow we have that*

$$\boldsymbol{\eta} = S \quad \text{for } \tau \equiv 0.$$

Then the dependence of the mobilities on the saturation function S *is given by* $\lambda_w(S)$ *and* $\lambda_n(S) \stackrel{def}{=} \lambda_n(1-S)$. *This corresponds to the well known mathematical model of the immiscible incompressible two-phase flow in porous medium considered by many authors (see, e.g., [2, 7, 8, 9, 11] and the references therein).*

Remark 3 *In some cases of the immiscible two-phase flow in a porous medium, like water-oil displacement, the local water saturation is increasing, or at least nondecreasing (see [6]), so that*

$$\frac{\partial S}{\partial t} \geq 0.$$

We consider here only such flows and hence we exclude from the consideration the changes of the relative permeabilities and capillary pressure caused by hysteresis effects. Then it holds $\eta = S + \tau\frac{\partial S}{\partial t} \geq S$ *and for any non-decreasing function* f *it is* $f(\eta) \geq f(S)$.

By using the definition of the effective saturation $\boldsymbol{\eta}$ we immediately rewrite the system (2.1) as an equivalent system of elliptic partial differential equations with respect to $\boldsymbol{\eta}$, with t being a parameter:

for a. e. $t \in [0,T]$,

$$\begin{cases} \Phi(\boldsymbol{\eta} - S) - \tau \,\mathrm{div}\, \big(K(x)\lambda_w(\boldsymbol{\eta})\nabla p_w\big) = 0 \quad \text{in } \Omega; \\ -\Phi(\boldsymbol{\eta} - S) - \tau \,\mathrm{div}\, \big(K(x)\lambda_n(\boldsymbol{\eta})\nabla p_n\big) = 0 \quad \text{in } \Omega; \\ P_c(\boldsymbol{\eta}) = p_n - p_w \quad \text{in } \Omega; \\ \boldsymbol{\eta} = S + \tau \dfrac{\partial S}{\partial t} \quad \text{in } \Omega. \end{cases} \tag{2.4}$$

The boundary conditions for the system (2.4) are given by (2.2).

In what follows we are going to use the following functions:

$$\alpha(s) \stackrel{\text{def}}{=} \frac{\lambda_w(s)\,\lambda_n(s)}{\lambda(s)}\,|P_c'(s)|; \quad \beta(s) \stackrel{\text{def}}{=} \int_0^s \alpha(\xi)\,d\xi;$$

$$a(s) \stackrel{\text{def}}{=} \sqrt{\frac{\lambda_w(s)\,\lambda_n(s)}{\lambda(s)}}\,|P_c'(s)|; \quad b(s) \stackrel{\text{def}}{=} \int_0^s a(\xi)\,d\xi. \tag{2.5}$$

In order to define the phase mobility functions as functions of variable $\boldsymbol{\eta}$, we first note the following.

i) There is a value of the saturation function S, S_0, such that

$$\boldsymbol{\eta} = 0 \quad \text{for} \quad S = S_0(x,t) = S^0(x)\,e^{-\frac{t}{\tau}}, \quad t > 0. \tag{2.6}$$

ii) There is a value of the saturation function S, S_1, such that

$$\boldsymbol{\eta} = 1 \quad \text{for} \quad S = S_1(x,t) = 1 + (S^0(x) - 1)\,e^{-\frac{t}{\tau}}, \quad t > 0. \tag{2.7}$$

Remark 4 *Provided* $0 \le S^0(x) \le 1$ *a. e. in* Ω*, from* (2.6)*,* (2.7) *we have* $0 \le S_0, S_1 \le 1$*. In this case a maximum principle for the actual saturation* S*,* $0 \le S \le 1$*, is a consequence of a maximum principle for the effective saturation* $\boldsymbol{\eta}$*,* $0 \le \boldsymbol{\eta} \le 1$*.*

Taking into account the standard assumptions on the equilibrium mobility functions (see the assumption (A.3) below), we define:

$$\lambda_w(\boldsymbol{\eta}) \stackrel{\text{def}}{=} \begin{cases} 0, \quad \text{when} \quad S < S_0 \quad \text{i.e.when} \quad \boldsymbol{\eta} < 0; \\ \lambda_w^e(\boldsymbol{\eta}), \text{when} \quad S_0 \le S \le S_1 \quad \text{i.e.when} \quad 0 \le \boldsymbol{\eta} \le 1; \\ 1, \quad \text{when} \quad S > S_1 \quad \text{i.e.when} \quad \boldsymbol{\eta} > 1 \end{cases} \tag{2.8}$$

and

$$\lambda_n(\boldsymbol{\eta}) \stackrel{\text{def}}{=} \begin{cases} 1, \quad \text{when} \quad S < S_0 \quad \text{i.e.when} \quad \boldsymbol{\eta} < 0; \\ \lambda_n^e(\boldsymbol{\eta}), \text{when} \quad S_0 \le S \le S_1 \quad \text{i.e.when} \quad 0 \le \boldsymbol{\eta} \le 1; \\ 0, \quad \text{when} \quad S > S_1 \quad \text{i.e.when} \quad \boldsymbol{\eta} > 1. \end{cases} \tag{2.9}$$

Here λ_w^e and λ_n^e denote the equilibrium mobility functions with standard properties. Our functions $\boldsymbol{\eta} \mapsto \lambda_w(\boldsymbol{\eta}), \lambda_n(\boldsymbol{\eta})$ share these properties.

2.2 Global pressure

In order to deal with the degeneracy in the system, we are going to use the concept of the global pressure, introduced in [2], [8] and standardly used since then. Namely, the idea is to induct a new pressure-like variable P, called the global pressure, which can be seen as a mixture pressure, where the two phases are considered as mixture constituents, of a flow which obeys Darcy law with a non-degenerate coefficient.

Following [2], [8], the global pressure P is defined by

$$p_w = P + G_w(\boldsymbol{\eta}),\ p_n = P + G_n(\boldsymbol{\eta}),$$

where

$$G_w(\boldsymbol{\eta}) = G_n(0) - \int_0^{\boldsymbol{\eta}} \frac{\lambda_n(\xi)}{\lambda(\xi)} P_c'(\xi)d\xi, \quad G_n(\boldsymbol{\eta}) = G_n(0) + \int_0^{\boldsymbol{\eta}} \frac{\lambda_w(\xi)}{\lambda(\xi)} P_c'(\xi)d\xi.$$

Then the following equalities hold:

$$\lambda_w(\boldsymbol{\eta})\, \nabla p_w + \lambda_n(\boldsymbol{\eta})\, \nabla p_n = \lambda(\boldsymbol{\eta})\, \nabla P,$$

$$\lambda_w(\boldsymbol{\eta})\, \nabla G_w(\boldsymbol{\eta}) = \alpha(\boldsymbol{\eta})\, \nabla \boldsymbol{\eta}, \quad \lambda_n(\boldsymbol{\eta})\, \nabla G_n(\boldsymbol{\eta}) = -\alpha(\boldsymbol{\eta})\, \nabla \boldsymbol{\eta}$$

and

$$\lambda_w(\boldsymbol{\eta})\, |\nabla p_w|^2 + \lambda_n(\boldsymbol{\eta})\, |\nabla p_n|^2 = \lambda(\boldsymbol{\eta})\, |\nabla P|^2 + |\nabla b(\boldsymbol{\eta})|^2. \tag{2.10}$$

2.3 Main assumptions

The main assumptions on the data are:

(A.1) The porosity Φ belongs to $L^\infty(\Omega)$, and there exist constants, $0 < \phi_m \le \phi_M < +\infty$, such that $0 < \phi_m \le \Phi(x) \le \phi_M$ a.e. in Ω.

(A.2) The permeability tensor K belongs to $(L^\infty(\Omega))^{d\times d}$, and there exist constants $0 < k_m \le k_M < +\infty$, such that for almost all $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbf{R}^d$ it holds:

$$k_m|\boldsymbol{\xi}|^2 \le K(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le k_M|\boldsymbol{\xi}|^2.$$

(A.3) The relative mobilities satisfy $\lambda_w, \lambda_n \in C([0,1]; \mathbf{R}^+)$, $\lambda_w(S_w = 0) = 0$ and $\lambda_n(S_n = 0) = 0$; λ_j is an increasing function of S_j. Moreover, there exist constants $\lambda_M \ge \lambda_m > 0$ such that for all $S \in [0,1]$

$$0 < \lambda_m \le \lambda_w(S) + \lambda_n(S) \le \lambda_M.$$

(A.4) The capillary pressure function $P_c \in C^1([0,1]; \mathbf{R}^+)$, $P_c' < 0$ in $[0,1]$ and $P_c(1) = 0$.

(A.5) The initial data satisfy $p_w^0, p_n^0 \in L^2(\Omega)$; $S^0 \in L^\infty(\Omega)$, $0 \le S^0 \le 1$ a.e. in Ω.

In order to formulate the results, let us introduce the following Sobolev space:

$$H^1_{\Gamma_1}(\Omega) \stackrel{\text{def}}{=} \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}.$$

The space $H^1_{\Gamma_1}(\Omega)$ is a Hilbert space when it is equipped with the norm $||u||_{H^1_{\Gamma_1}(\Omega)} = ||\nabla u||_{(L^2(\Omega))^d}$.

Our interest concerns the existence of the weak solution of the system (2.1), (2.2), (2.3), or equivalently, the existence for the elliptic system (2.4), (2.2).

2.4 Energy equality and formal a priori estimates

By formally multiplying the first equation in (2.1) by p_w, the second equation in (2.1) by p_n, integrating over Ω_T and summing the two equations we obtain the "energy equality"

$$-\int_{\Omega_T} \Phi \partial_t S \, P_c(\boldsymbol{\eta}) dx dt + \int_{\Omega_T} K(x) \, \left(\lambda_w(\boldsymbol{\eta}) \, \nabla p_w \cdot \nabla p_w + \lambda_n(\boldsymbol{\eta}) \, \nabla p_n \cdot \nabla p_n\right) dx dt = 0. \tag{2.11}$$

Denote the second integral in (2.11) by I, then

$$I = \int_{\Omega_T} \Phi \partial_t S \cdot \, P_c(\boldsymbol{\eta}) dx dt = \frac{1}{\tau} \int_{\Omega_T} \Phi(\boldsymbol{\eta} - S) \, P_c(\boldsymbol{\eta}) dx dt. \tag{2.12}$$

From $\boldsymbol{\eta} = S + \tau \, \frac{\partial S}{\partial t}$ we can express

$$S(x,t) = S^0(x) \, \exp(-\frac{t}{\tau}) + \frac{1}{\tau} \, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta})(x,t). \tag{2.13}$$

Here $\star$ denotes convolution with respect to time. Then

$$\frac{\partial S}{\partial t}(x,t) = -\frac{1}{\tau} \, S^0(x) \, \exp(-\frac{t}{\tau}) - \frac{1}{\tau^2} \, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta})(x,t) + \frac{1}{\tau} \, \boldsymbol{\eta}(x,t)$$

and

$$I = \frac{1}{\tau} \int_{\Omega_T} \Phi \, \Big(- S^0(x) \, \exp(-\frac{t}{\tau}) - \frac{1}{\tau} \, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta})(x,t) + \boldsymbol{\eta}(x,t)) \Big) \cdot \, P_c(\boldsymbol{\eta}) dx dt.$$

We write

$$I = -\frac{1}{\tau} \int_{\Omega_T} \Phi \, S^0(x) \, \exp(-\frac{t}{\tau}) \cdot \, P_c(\boldsymbol{\eta}) dx dt - \frac{1}{\tau^2} \int_{\Omega_T} \Phi \, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta})(x,t) \cdot \, P_c(\boldsymbol{\eta}) dx dt +$$

$$+ \frac{1}{\tau} \int_{\Omega_T} \Phi \, \boldsymbol{\eta}(x,t) \cdot \, P_c(\boldsymbol{\eta}) dx dt = I_1 + I_2 + I_3.$$

Then by using (A.1), (A.4) and (A.5) we obtain for the first term:

$$I_1 \leq \frac{1}{\tau} \, \phi_M \, P_{c,max} \, |\Omega_T|.$$

Further, by using $(\mathsf{A.1})$, $(\mathsf{A.4})$, boundedness of the exponential function and $0 \leq \boldsymbol{\eta} \leq 1$, we get for the second term:

$$I_2 \leq \frac{1}{\tau^2}\,\phi_M\,P_{c,max}\int_{\Omega_T}|\exp(-\frac{t}{\tau})\star\boldsymbol{\eta}(x,t)|dx$$

$$\leq \frac{1}{\tau^2}\,\phi_M\,P_{c,max}\int_{\Omega}\|\exp(-\frac{t}{\tau})\|_{L^1([0,T])}\cdot\|\boldsymbol{\eta}(x,t)\|_{L^1([0,T])}dx \leq$$

$$\leq \frac{1}{\tau^2}\,\phi_M\,P_{c,max}\,\tau\,(1-\exp(-\frac{T}{\tau}))\,|\Omega_T| \leq \frac{1}{\tau}\,\phi_M\,P_{c,max}\,|\Omega_T|.$$

Finally, the third term can be bounded by using $(\mathsf{A.1})$, $(\mathsf{A.4})$ and $0 \leq \boldsymbol{\eta} \leq 1$ as follows:

$$I_3 \leq \frac{1}{\tau}\,\phi_M\,P_{c,max}\,|\Omega_T|.$$

Therefore

$$I \leq C$$

and from (2.11) and the equality (2.10) we obtain a priori estimates

$$\|\sqrt{\lambda_w(\boldsymbol{\eta})}\nabla p_w\|_{L^2(\Omega_T)} + \|\sqrt{\lambda_n(\boldsymbol{\eta})}\nabla p_n\|_{L^2(\Omega_T)} + \|P\|_{L^2([0,T];H^1(\Omega))} + \|b(\boldsymbol{\eta})\|_{L^2([0,T];H^1(\Omega))} \leq C. \tag{2.14}$$

From (2.14) and the system under consideration then it easily follows

$$\left\|\Phi\frac{\partial S}{\partial t}\right\|_{L^2([0,T];H^{-1}(\Omega))} \leq C. \tag{2.15}$$

Remark 5 *The expression* (2.13) *reveals that the function* $S(x,t)$, $t\in[0,T]$, $x\in\Omega$, *contains the full history of* $\boldsymbol{\eta}$, *namely the values* $\boldsymbol{\eta}(x,\xi), 0\leq\xi\leq t$.

3 Regularized system

3.1 Main result

In order to prove the desired existence result, we introduce a small positive parameter θ and form an auxiliary regularized problem. This is performed by adding to the system a saturation term $\pm\theta\,\mathrm{div}\,\nabla P_c(\boldsymbol{\eta})$, preserving a maximum principle (see [12], [1]). This gives us the following system, parametrized by θ:

$$\begin{cases} \Phi\dfrac{\partial S^\theta}{\partial t} - \mathrm{div}\left(K(x)\lambda_w(\boldsymbol{\eta}^\theta)\nabla p_w^\theta\right) + \theta\,\mathrm{div}\,\nabla P_c(\boldsymbol{\eta}^\theta) = 0 \quad \text{in } \Omega_T, \\ -\Phi\dfrac{\partial S^\theta}{\partial t} - \mathrm{div}\left(K(x)\lambda_n(\boldsymbol{\eta}^\theta)\nabla p_n^\theta\right) - \theta\,\mathrm{div}\,\nabla P_c(\boldsymbol{\eta}^\theta) = 0 \quad \text{in } \Omega_T, \\ P_c(\boldsymbol{\eta}^\theta) = p_n^\theta - p_w^\theta, \\ \boldsymbol{\eta}^\theta = S^\theta + \tau\,\dfrac{\partial S^\theta}{\partial t}. \end{cases} \tag{3.1}$$

The corresponding initial conditions are given by (2.3) (with index θ). The corresponding boundary conditions are

$$\begin{cases} p_w^\theta(x,t) = p_n^\theta(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,T); \\ \big(\mathbf{W}_w - \theta\,\nabla P_c(\boldsymbol{\eta}^\theta)\big)\cdot\vec{\nu} = \big(\mathbf{W}_n + \theta\,\nabla P_c(\boldsymbol{\eta}^\theta)\big)\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_2 \times (0,T). \end{cases} \tag{3.2}$$

The equivalent regularized elliptic system is given for a. e. $t \in [0,T]$ by

$$\begin{cases} \Phi(\boldsymbol{\eta}^\theta - S^\theta) - \tau\,\mathrm{div}\,\big(K(x)\lambda_w(\boldsymbol{\eta}^\theta)\nabla p_w^\theta\big) + \tau\,\theta\,\mathrm{div}\,\nabla P_c(\boldsymbol{\eta}^\theta) = 0 \quad \text{in } \Omega, \\ -\Phi(\boldsymbol{\eta}^\theta - S^\theta) - \tau\,\mathrm{div}\,\big(K(x)\lambda_n(\boldsymbol{\eta}^\theta)\nabla p_n^\theta\big) - \tau\,\theta\,\mathrm{div}\,\nabla P_c(\boldsymbol{\eta}^\theta) = 0 \quad \text{in } \Omega, \\ P_c(\boldsymbol{\eta}^\theta) = p_n^\theta - p_w^\theta \quad \text{in } \Omega, \\ \boldsymbol{\eta}^\theta = S^\theta + \tau\,\dfrac{\partial S^\theta}{\partial t}. \end{cases} \tag{3.3}$$

Boundary conditions for the system (3.3) are given by (3.2).

Our main result concerns the existence of the weak solutions of the system (3.3) and is given in the following Theorem.

Theorem 3.1 *Let assumptions* (*A.1*)-(*A.5*) *be fulfilled and let* $0 < \theta \ll 1$. *Then there exist* $p_w^\theta, p_n^\theta, S^\theta, \boldsymbol{\eta}^\theta$ *such that for a. e.* $t \in [0,T]$

- $p_w^\theta(t), p_n^\theta(t) \in H^1_{\Gamma_1}(\Omega)$, $S^\theta(t) \in H^1(\Omega)$;
- $0 \le \boldsymbol{\eta}^\theta(t) \le 1$ *a. e. in* Ω, $0 \le S^\theta(t) \le 1$ *a. e. in* Ω;
- *For all* $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(\boldsymbol{\eta}^\theta - S^\theta)\varphi_w dx + \tau \int_\Omega K(x)\,\lambda_w(\boldsymbol{\eta}^\theta)\,\nabla p_w^\theta \cdot \nabla\varphi_w dx - \tau\theta \int_{\Omega_T} \nabla P_c(\boldsymbol{\eta}^\theta)\cdot\nabla\varphi_w dxdt = 0, \tag{3.4}$$

$$-\int_\Omega \Phi(\boldsymbol{\eta}^\theta - S^\theta)\varphi_n dx + \tau \int_\Omega K(x)\,\lambda_n(\boldsymbol{\eta}^\theta)\,\nabla p_n^\theta \cdot \nabla\varphi_n dx + \tau\theta \int_{\Omega_T} \nabla P_c(\boldsymbol{\eta}^\theta)\cdot\nabla\varphi_n dxdt = 0. \tag{3.5}$$

The proof of Theorem 3.1 is achieved in several steps. First we introduce two regularizations, with respect to parameters ε (Subsection 3.2) and N (Subsection 3.3). In Subsection 3.4 we obtain the existence for the ε, N - problem. In the following Subsection 3.5 we derive uniform estimates for the solutions with respect to N, conclude about the consequent convergence results and pass to the limit as $N \to \infty$ in the ε, N - problem. Finally, in Subsection 3.6 we establish the uniform estimates in ε, and pass to the limit in the ε - problem by using the obtained convergence results for the solutions.

3.2 Introduction of a small parameter $\varepsilon > 0$

In order to achieve uniform ellipticity of the system, a small constant is added to the mobility functions (as in [12], [1]). Accordingly, let us introduce the non-degenerate mobility functions λ_w^ε, λ_n^ε defined by

$$\lambda_w^\varepsilon \stackrel{\text{def}}{=} \lambda_w + \varepsilon, \quad \lambda_n^\varepsilon \stackrel{\text{def}}{=} \lambda_n + \varepsilon, \quad 0 < \varepsilon \ll 1. \tag{3.6}$$

We will hence consider the system (3.3) with λ_w^ε, λ_n^ε instead of λ_w, λ_n. Using such non-degenerate mobilities will result in the loss of the maximum principle for the effective saturation $\boldsymbol{\eta}$ and therefore we introduce the following continuous extensions of our functions outside of $[0, 1]$:

$$Z(\boldsymbol{\eta}) = \begin{cases} 0 \text{ for } \boldsymbol{\eta} \le 0; \\ \boldsymbol{\eta} \text{ for } 0 \le \boldsymbol{\eta} \le 1; \\ 1 \text{ for } \boldsymbol{\eta} \ge 0. \end{cases} \qquad \overline{\lambda}_w(\boldsymbol{\eta}) = \begin{cases} 0 \text{ for } \boldsymbol{\eta} \le 0; \\ \lambda_w(\boldsymbol{\eta}) \text{ for } 0 \le \boldsymbol{\eta} \le 1; \\ \lambda_w(1) \text{ for } \boldsymbol{\eta} \ge 1. \end{cases}$$

$$\overline{\lambda}_n(\boldsymbol{\eta}) = \begin{cases} \lambda_n(1) \text{ for } \boldsymbol{\eta} \le 0; \\ \lambda_n(\boldsymbol{\eta}) \text{ for } 0 \le \boldsymbol{\eta} \le 1; \\ 0 \text{ for } \boldsymbol{\eta} \ge 1. \end{cases} \qquad \overline{P}_c(\boldsymbol{\eta}) = \begin{cases} P_c(0) - \boldsymbol{\eta} \text{ for } \boldsymbol{\eta} \le 0; \\ P_c(\boldsymbol{\eta}) \text{ for } 0 \le \boldsymbol{\eta} \le 1; \\ 1 - \boldsymbol{\eta} \text{ for } \boldsymbol{\eta} \ge 1. \end{cases}$$

Such extending is possible due to the assumptions (A.3) and (A.4). All extensions are bounded functions on $\mathbb{R}$ except $\overline{P_c}(\boldsymbol{\eta})$ (but $\overline{P_c}(\boldsymbol{\eta})'$ is bounded).
We note that now we can express the saturations as functions of the principle unknowns p_w^θ, p_n^θ:

$$\boldsymbol{\eta}^\theta = \overline{P}_c^{-1}(p_n^\theta - p_w^\theta)$$

and

$$S^\theta(x,t) = S^0(x)\, \exp(-\frac{t}{\tau}) + \frac{1}{\tau}\,(\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta}^\theta)(x,t).$$

For notational convenience, we will skip the solution subscript θ in what follows. Now we consider the system (3.3) with the extended saturations and the extended non-degenerate mobilities, i. e. for a. e. $t \in [0,T]$ we have the system

$$\begin{cases} \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon)) - \tau \operatorname{div}\left(K(x)\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta}^\varepsilon)\nabla p_w^\varepsilon\right) + \tau\,\theta \operatorname{div}\nabla P_c(\boldsymbol{\eta}^\varepsilon) = 0 \quad \text{in } \Omega; \\ -\Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon)) - \tau \operatorname{div}\left(K(x)\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta}^\varepsilon)\nabla p_n^\varepsilon\right) - \tau\,\theta \operatorname{div}\nabla P_c(\boldsymbol{\eta}^\varepsilon) = 0 \quad \text{in } \Omega; \\ P_c(\boldsymbol{\eta}^\varepsilon) = p_n^\varepsilon - p_w^\varepsilon \quad \text{in } \Omega; \\ \boldsymbol{\eta}^\varepsilon = S^\varepsilon + \tau\,\dfrac{\partial S^\varepsilon}{\partial t}, \end{cases} \tag{3.7}$$

i. e. for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon))\varphi_w dx + \tau \int_\Omega K(x)\,\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta}^\varepsilon)\,\nabla p_w^\varepsilon \cdot \nabla\varphi_w dx - \tau\theta \int_\Omega \nabla\overline{P}_c(\boldsymbol{\eta}^\varepsilon)\cdot\nabla\varphi_w dx = 0, \tag{3.8}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon))\varphi_n dx + \tau \int_\Omega K(x)\,\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta}^\varepsilon)\,\nabla p_n^\varepsilon \cdot \nabla\varphi_n dx + \tau\theta \int_\Omega \nabla\overline{P}_c(\boldsymbol{\eta}^\varepsilon)\cdot\nabla\varphi_n dx = 0. \tag{3.9}$$

3.3 Introduction of parameter $N \in \mathbb{N}$

Following [12], [1], we introduce also another regularization. Namely, for $N \in \mathbb{N}$ we denote by $\mathcal{P}_N$ the orthogonal projector of $L^2(\Omega)$ on the first N eigenvectors of the Laplace operator with homogeneous Dirichlet boundary conditions. We replace p_w^ε, p_n^ε in the additional terms in (3.7) by $\mathcal{P}_N[p_w^\varepsilon]$, $\mathcal{P}_N[p_n^\varepsilon]$ (in other words, by their orthogonal projections on finite-dimensional subspaces defined in terms of the eigenbasis of the Laplace operator in Ω with Dirichlet boundary conditions) and for a. e. $t \in [0,T]$ consider the following elliptic system in Ω:

$$\begin{cases} \Phi(Z(\boldsymbol{\eta}^{\varepsilon,N}) - Z(S^{\varepsilon,N})) - \tau \operatorname{div}\left(K(x)\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta}^{\varepsilon,N})\nabla p_w^{\varepsilon,N}\right) + \tau\,\theta \operatorname{div}\nabla\left(\mathcal{P}_N[p_n^{\varepsilon,N}] - \mathcal{P}_N[p_w^{\varepsilon,N}]\right) = 0; \\ -\Phi(Z(\boldsymbol{\eta}^{\varepsilon,N}) - Z(S^{\varepsilon,N})) - \tau \operatorname{div}\left(K(x)\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta}^{\varepsilon,N})\nabla p_n^{\varepsilon,N}\right) - \tau\,\theta \operatorname{div}\nabla\left(\mathcal{P}_N[p_n^{\varepsilon,N}] - \mathcal{P}_N[p_w^{\varepsilon,N}]\right) = 0; \\ P_c(\boldsymbol{\eta}^{\varepsilon,N}) = p_n^{\varepsilon,N} - p_w^{\varepsilon,N}; \\ \boldsymbol{\eta}^{\varepsilon,N} = S^{\varepsilon,N} + \tau\,\dfrac{\partial S^{\varepsilon,N}}{\partial t} \end{cases} \tag{3.10}$$

i. e. for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}^{\varepsilon,N}) - Z(S^{\varepsilon,N}))\varphi_w dx + \tau \int_\Omega K(x)\,\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta}^{\varepsilon,N})\,\nabla p_w^{\varepsilon,N}\cdot\nabla\varphi_w dx - \\ -\tau\theta\int_\Omega \nabla\left(\mathcal{P}_N[p_n^{\varepsilon,N}] - \mathcal{P}_N[p_w^{\varepsilon,N}]\right)\cdot\nabla\varphi_w dx = 0, \tag{3.11}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}^{\varepsilon,N}) - Z(S^{\varepsilon,N}))\varphi_n dx + \tau \int_\Omega K(x)\,\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta}^{\varepsilon,N})\,\nabla p_n^{\varepsilon,N}\cdot\nabla\varphi_n dx + \\ +\tau\theta\int_\Omega \nabla\left(\mathcal{P}_N[p_n^{\varepsilon,N}] - \mathcal{P}_N[p_w^{\varepsilon,N}]\right)\cdot\nabla\varphi_n dx = 0. \tag{3.12}$$

3.4 Existence for the θ, ε, N - problem

For fixed $N > 0$ and $\varepsilon > 0$ now we show the existence of solutions to the system (3.11)-(3.12) by applying the Leray-Schauder fixed point theorem. We quote the result here.

Theorem 3.2 *Let $\mathcal{T}$ be a continuous and compact map of a Banach space $\mathcal{B}$ into itself. Suppose that the set $\{x \in \mathcal{B} : x = \sigma\mathcal{T}x$ for some $\sigma \in [0,1]\}$ is bounded. Then the map $\mathcal{T}$ has a fixed point.*

The proof follows arguments of [12] (see also [1], [10]). Let us rewrite the system (3.11)-(3.12) for fixed $\theta > 0$, $\varepsilon > 0$ and $N \in \mathbb{N}$ but, for notational convenience, without the dependence of the solutions on the parameters θ, ε and N: for a. e. $t \in [0,T]$; for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S)\varphi_w dx + \tau\int_\Omega K(x)\,\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta})\,\nabla p_w\cdot\nabla\varphi_w dx - \tau\theta\int_\Omega \nabla\left(\mathcal{P}_N[p_n] - \mathcal{P}_N[p_w]\right)\cdot\nabla\varphi_w dx = 0, \tag{3.13}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S))\varphi_n dx + \tau\int_\Omega K(x)\,\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta})\,\nabla p_n\cdot\nabla\varphi_n dx + \tau\theta\int_\Omega \nabla\left(\mathcal{P}_N[p_n] - \mathcal{P}_N[p_w]\right)\cdot\nabla\varphi_n dx = 0. \tag{3.14}$$

Now consider a map $\mathcal{T} : (L^2(\Omega))^2 \to (L^2(\Omega))^2$ defined by

$$\mathcal{T}(\overline{p}_w, \overline{p}_n) = (p_w, p_n), \tag{3.15}$$

where the pair (p_w, p_n) is the unique solution of the following system: for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(Z(\overline{\boldsymbol{\eta}}) - Z(\overline{S})\varphi_w dx + \tau \int_\Omega K(x)\, \overline{\lambda}^\varepsilon_w(\overline{\boldsymbol{\eta}})\, \nabla p_w \cdot \nabla \varphi_w dx - \tau\theta \int_\Omega \nabla\left(\mathcal{P}_N[\overline{p}_n] - \mathcal{P}_N[\overline{p}_w]\right) \cdot \nabla \varphi_w dx = 0, \tag{3.16}$$

$$-\int_\Omega \Phi(Z(\overline{\boldsymbol{\eta}}) - Z(\overline{S}))\varphi_n dx + \tau \int_\Omega K(x)\, \overline{\lambda}^\varepsilon_n(\overline{\boldsymbol{\eta}})\, \nabla p_n \cdot \nabla \varphi_n dx + \tau\theta \int_\Omega \nabla\left(\mathcal{P}_N[\overline{p}_n] - \mathcal{P}_N[\overline{p}_w]\right) \cdot \nabla \varphi_n dx = 0. \tag{3.17}$$

Here

$$\overline{\boldsymbol{\eta}} = \overline{P_c}^{-1}(\overline{p}_n - \overline{p}_w)$$

and

$$\overline{S}(x,t) = S^0(x)\, \exp(-\frac{t}{\tau}) + \frac{1}{\tau}\,(\exp(-\frac{t}{\tau}) \star \overline{\boldsymbol{\eta}})(x,t).$$

Denote

$$B_w(p_w, \varphi_w) \stackrel{\text{def}}{=} \tau \int_\Omega K(x)\, \overline{\lambda^\varepsilon_w}(\boldsymbol{\eta})\, \nabla p_w \cdot \nabla \varphi_w dx;$$

$$B_n(p_n, \varphi_n) \stackrel{\text{def}}{=} \tau \int_\Omega K(x)\, \overline{\lambda^\varepsilon_n}(\boldsymbol{\eta})\, \nabla p_n \cdot \nabla \varphi_n dx;$$

$$f_w(\varphi_w) \stackrel{\text{def}}{=} -\int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S)\varphi_w dx + \tau\theta \int_\Omega \nabla\left(\mathcal{P}_N[p_n] - \mathcal{P}_N[p_w]\right) \cdot \nabla \varphi_w dx;$$

$$f_n(\varphi_n) \stackrel{\text{def}}{=} \int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S)\varphi_n dx - \tau\theta \int_\Omega \nabla\left(\mathcal{P}_N[p_n] - \mathcal{P}_N[p_w]\right) \cdot \nabla \varphi_n dx.$$

The system (3.13)-(3.14) can be seen as

$$B_w(p_w, \varphi_w) = f_w(\varphi_w); \quad B_n(p_n, \varphi_n) = f_n(\varphi_n) \text{ for all } \varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega).$$

The maps B_w and B_n are clearly bilinear, continuous and coercive on $(H^1_{\Gamma_1}(\Omega))^2$ (the coercivity is in particular a consequence of the regularization (3.6)). The maps $f_w(\varphi_w)$, $f_n(\varphi_n)$ are clearly linear and continuous on $H^1_{\Gamma_1}(\Omega)$. Therefore, we can apply Lax-Milgram theorem to obtain the existence of the unique pair $(p_w, p_n) \in (H^1_{\Gamma_1}(\Omega))^2$, for all $(\overline{p}_w, \overline{p}_n) \in (L^2(\Omega))^2$, i. e. the map $\mathcal{T}$ is well defined. Further, it is shown that $\mathcal{T}$ is relatively compact and continuous, as well as that $\{x \in \mathcal{B} : x = \sigma \mathcal{T} x \text{ for some } \sigma \in [0,1]\}$ is bounded, by using the arguments of [1], [10]. Hence, the following Proposition is proved.

Proposition 3.3 *For fixed θ, ε and N there is a weak solution of the system* (3.10) *for a. e. $t \in [0, T]$.*

3.5 Uniform estimates in N and passing to the limit as $N \to \infty$

In this part for simplicity we skip the dependence of the solutions on ε. For all N in Subsection 3.4 we have obtained for a. e. $t \in [0, T]$ a solution $(p_w, p_n) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ of the following system: for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}^N) - Z(S^N))\varphi_w dx + \tau \int_\Omega K(x)\, \overline{\lambda}^\varepsilon_w(\boldsymbol{\eta}^N)\, \nabla p^N_w \cdot \nabla \varphi_w dx - \\ -\tau\theta \int_\Omega \nabla \left(\mathcal{P}_N[p^N_n] - \mathcal{P}_N[p^N_w]\right) \cdot \nabla \varphi_w dx dt = 0, \tag{3.18}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}^N) - Z(S^N))\varphi_n dx + \tau \int_\Omega K(x)\, \overline{\lambda}^\varepsilon_n(\boldsymbol{\eta}^N)\, \nabla p^N_n \cdot \nabla \varphi_n dx + \\ +\tau\theta \int_\Omega \nabla \left(\mathcal{P}_N[p^N_n] - \mathcal{P}_N[p^N_w]\right) \cdot \nabla \varphi_n dx dt = 0. \tag{3.19}$$

By choosing test functions $\varphi_w = p^N_w \in H^1_{\Gamma_1}(\Omega)$ in (3.18) and $\varphi_n = p^N_n \in H^1_{\Gamma_1}(\Omega)$ in (3.19) and summing the two equations we obtain the following equality:

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}^N) - Z(S^N)) \cdot (p^N_n - p^N_w)\, dx + \tau \int_\Omega K(x) \left(\overline{\lambda}^\varepsilon_w(\boldsymbol{\eta}^N)\, \nabla p^N_w \cdot \nabla p^N_w + \overline{\lambda}^\varepsilon_n(\boldsymbol{\eta}^N)\, \nabla p^N_n \cdot \nabla p^N_n\right) dx \\ +\tau\theta \int_\Omega \nabla \left(\mathcal{P}_N[p^N_n] - \mathcal{P}_N[p^N_w]\right) \cdot \nabla \left(p^N_n - p^N_w\right) dx dt = 0, \tag{3.20}$$

which leads to the following estimate, uniform in N:

$$\varepsilon\, \tau \int_\Omega \left(|\nabla p^N_w|^2 + |\nabla p^N_n|^2\right)\, dx + \theta\, \tau \int_\Omega |\nabla(\mathcal{P}_N[p^N_n] - \mathcal{P}_N[p^N_w])|^2\, dx \le C. \tag{3.21}$$

Therefore, from the inequality (3.21) and by taking into account that $\boldsymbol{\eta}^N = \overline{P}_c^{-1}(p^N_n - p^N_w)$ and

$$S^N(x,t) = S^0(x)\, \exp(-\frac{t}{\tau}) + \frac{1}{\tau}\, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta}^N)(x,t),$$

up to subsequences, we have the following convergence results as $N \to \infty$:

$$\begin{aligned} p^N_w &\to p^\varepsilon_w \text{ weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega), \text{ a. e. in } \Omega; \\ p^N_n &\to p^\varepsilon_n \text{ weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega), \text{ a. e. in } \Omega; \\ \boldsymbol{\eta}^N &\to \boldsymbol{\eta}^\varepsilon \text{ strongly in } L^2(\Omega), \text{ a. e. in } \Omega; \\ S^N &\to S^\varepsilon \stackrel{\text{def}}{=} S^0(x)\, \exp(-\frac{t}{\tau}) + \frac{1}{\tau}\, (\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta}^\varepsilon)(x,t) \text{ a. e. in } \Omega. \end{aligned}$$

Now we can pass to the limit as $N \to \infty$ in the system (3.18), (3.19) to obtain for all $\varphi_w, \varphi_n \in H^1_{\Gamma_1}(\Omega)$ (again we denote the dependence on ε):

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon))\varphi_w dx + \tau \int_\Omega K(x)\, \overline{\lambda}^\varepsilon_w(\boldsymbol{\eta}^\varepsilon)\, \nabla p^\varepsilon_w \cdot \nabla \varphi_w dx - \tau\theta \int_{\Omega_T} \nabla\, (p^\varepsilon_n - p^\varepsilon_w) \cdot \nabla \varphi_w dx dt = 0, \tag{3.22}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon))\varphi_n dx + \tau \int_\Omega K(x)\,\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta}^\varepsilon)\,\nabla p_n^\varepsilon \cdot \nabla\varphi_n dx + \tau\theta \int_{\Omega_T} \nabla\left(p_n^\varepsilon - p_w^\varepsilon\right) \cdot \nabla\varphi_n dx dt = 0. \tag{3.23}$$

The passage to the limit in the term $\int_\Omega \Phi(Z(\boldsymbol{\eta}^\varepsilon) - Z(S^\varepsilon))\varphi_w\, dx$ is achieved by Lebesgue dominated convergence theorem.

3.6 Uniform estimates in ε and passing to the limit as $\varepsilon \to 0$

From the results of the previous Subsection it follows that for any $\varepsilon > 0$ there is $(p_w^\varepsilon, p_n^\varepsilon) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, a solution of the system (3.22)-(3.23). We choose $\varphi_w = p_w^\varepsilon$ in (3.22) and $\varphi_n = p_n^\varepsilon$ in (3.23), sum the two equations and estimate analogously to the technique used in the previous Subsection, to prove the following Lemma.

Lemma 3.1 *Let $(p_w^\varepsilon, p_n^\varepsilon)$ be a solution to* (3.22)-(3.23) *and let P^ε be the corresponding global pressure. Then the following estimates hold, uniform in ε:*

$\{p_w^\varepsilon\}_\varepsilon$ *is uniformly bounded in* $H^1_{\Gamma_1}(\Omega)$, $\{p_n^\varepsilon\}_\varepsilon$ *is uniformly bounded in* $H^1_{\Gamma_1}(\Omega)$;
$\{P^\varepsilon\}_\varepsilon$ *is uniformly bounded in* $H^1(\Omega)$, $\{b(\boldsymbol{\eta}^\varepsilon)\}_\varepsilon$ *is uniformly bounded in* $H^1(\Omega)$;
$\{\overline{P}_c^\varepsilon\}_\varepsilon$ *is uniformly bounded in* $L^2(\Omega)$.

From Lemma 3.1 we conclude that the following convergence results hold as $\varepsilon \to 0$:

$$\begin{aligned}
p_w^\varepsilon &\to p_w \text{ weakly in } H^1(\Omega), \text{ a. e. in } \Omega;\\
p_n^\varepsilon &\to p_n \text{ weakly in } H^1(\Omega), \text{ a. e. in } \Omega;\\
P^\varepsilon &\to P \text{ weakly in } H^1(\Omega), \text{ a. e. in } \Omega;\\
b(\boldsymbol{\eta}^\varepsilon) &\to b(\boldsymbol{\eta}) \text{ weakly in } H^1(\Omega), \text{ a. e. in } \Omega;\\
Z(\boldsymbol{\eta}^\varepsilon) &\to Z(\boldsymbol{\eta}) \text{ strongly in } L^2(\Omega), \text{ a. e. in } \Omega;\\
Z(S^\varepsilon) \to Z(S) &\stackrel{\text{def}}{=} Z(S^0\, \exp(-\frac{t}{\tau}) + \frac{1}{\tau}\,(\exp(-\frac{t}{\tau}) \star \boldsymbol{\eta})) \text{ a. e. in } \Omega.
\end{aligned}$$

Now we pass to the limit as $\varepsilon \to 0$ in the system (3.22), (3.23) based on the obtained convergence results with respect to ε. This leads to the following system:

$$\int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S))\varphi_w dx + \tau \int_\Omega K(x)\,\overline{\lambda}_w^\varepsilon(\boldsymbol{\eta})\,\nabla p_w \cdot \nabla\varphi_w dx - \tau \int_{\Omega_T} \nabla\left(p_n - p_w\right) \cdot \nabla\varphi_w dx dt = 0, \tag{3.24}$$

$$-\int_\Omega \Phi(Z(\boldsymbol{\eta}) - Z(S))\varphi_n dx + \tau \int_\Omega K(x)\,\overline{\lambda}_n^\varepsilon(\boldsymbol{\eta})\,\nabla p_n \cdot \nabla\varphi_n dx + \tau \int_{\Omega_T} \nabla\left(p_n - p_w\right) \cdot \nabla\varphi_n dx dt = 0. \tag{3.25}$$

In order to finish the proof of Theorem 3.1, let us prove the maximum principle for the system (3.24) - (3.25).

Lemma 3.2 *Under the assumptions* ($A.1$)-($A.5$) *it holds*

$$0 \leq \boldsymbol{\eta} \leq 1 \text{ a. e. in } \Omega.$$

Proof of Lemma 3.2

First we will show that $\boldsymbol{\eta} \geq 0$. Let us introduce the following test function in (3.24):

$$\boldsymbol{\eta}^- = \min\{0, \boldsymbol{\eta}\} = \begin{cases} \boldsymbol{\eta} \text{ for } \boldsymbol{\eta} \leq 0; \\ 0 \text{ for } \boldsymbol{\eta} \geq 0. \end{cases}$$

Note that by the definition of these functions it is $\boldsymbol{\eta}^- \leq 0$ and $Z(\boldsymbol{\eta}) \cdot \boldsymbol{\eta}^- = 0$, $\overline{\lambda}_w(\boldsymbol{\eta}) \cdot \boldsymbol{\eta}^- = 0$, $\overline{\lambda}_n(\boldsymbol{\eta}) \cdot \boldsymbol{\eta}^- = 0$.

After multiplying the first equation in (3.24) by $\varphi_w = \boldsymbol{\eta}^-$ and integrating over x, by using the definition of the extended functions and the boundary conditions, we obtain:

$$\tau\,\theta \int_\Omega (\nabla \boldsymbol{\eta}^-)^2 dx \leq 0$$

which implies that $\boldsymbol{\eta}^- = 0$, that is, $\boldsymbol{\eta} \geq 0$ a. e. in Ω. In an analogous way it is proved that $\boldsymbol{\eta} \leq 1$ a. e. in Ω. Lemma 3.2 is proven. This completes the proof of Theorem 3.1.

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