ε-NEIGHBORHOODS OF ORBITS OF PARABOLIC DIFFEOMORPHISMS AND COHOMOLOGICAL EQUATIONS

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ABSTRACT. In this article, we study analyticity properties of (directed) areas of ε-neighborhoods of orbits of parabolic germs. The article is motivated by the question of analytic classification using ε-neighborhoods of orbits in the simplest formal class.

We show that the coefficient in front of ε^2 term in the asymptotic expansion in ε, which we call the principal part of the area, is a sectorially analytic function in the initial point of the orbit. It satisfies a cohomological equation similar to the standard trivialization equation for parabolic diffeomorphisms.

We give necessary and sufficient conditions on a diffeomorphism f for the existence of globally analytic solution of this equation. Furthermore, we introduce new classification type for diffeomorphisms implied by this new equation and investigate the relative position of its classes with respect to the analytic classes.

Keywords: parabolic diffeomorphisms, classification, cohomological equation, Stokes phenomenon, epsilon-neighborhoods of orbits
MSC 2010: 37C05, 37C10, 37C15, 34M40, 39A45, 40G10

1. Introduction and main results

1.1. **Motivation.** Each germ of a parabolic diffeomorphism in the complex plane,

$$f(z) = z + a_1 z^{k+1} + a_2 z^{k+2} + o(z^{k+2}),\ k \in \mathbb{N},\ a_i \in \mathbb{C},\ a_1 \neq 0, \tag{1}$$

can, by formal changes of variables, be reduced to the formal normal form, which is the time-one map of the holomorphic vector field

$$f_0(z) = \exp(X_{k,\rho}),\ X_{k,\rho} = \frac{z^{k+1}}{1 + \frac{\rho}{2\pi i} z^k} \frac{d}{dz},$$

for $k \in \mathbb{N}$ as in (1) and an appropriate $\rho \in \mathbb{C}$. Here, k is the same as in (1) and ρ depends on k and the first $k+1$ coefficients $a_1, \ldots, a_{k+1}$. The

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formal class of a parabolic germ is given by the pair (k, ρ), $k \in \mathbb{N}$, $\rho \in \mathbb{C}$. On the other hand, the analytic class is given by $2k$ diffeomorphisms on the spaces of closed orbits of the neighboring petals. They are called the *Écalle-Voronin moduli* or *horn maps*, see e.g. [6], [27] or [15].

For an overview on use of *fractal dimensions* in dynamical systems, see e.g.[28]. The most commonly used fractal dymensions are Hausdorff and *box dimension*. The exact definitions may be looked up in e.g. [26]. By [19], Hausdorff dimensions of strange attractors (Lorenz attractor, Smale horseshoe, Hénon attractor) reveal their complexity. Since Hausdorff dimension is in some cases trivial and not interesting due to its countable stability property, the box dimension may be used instead. Applied to appropriate invariant sets, they show intrinsic properties of dynamical systems. It was thus shown that the box dimension and the Minkowski content of only one trajectory of a discrete or a continuous dynamical system classify the system. One can read from them the multiplicity of the generating function of the discrete system, the moment or the complexity of bifurcation, see [8], [17], [29].

Along the same lines was the question if we can recognize a complex germ (1) using box dimension and Minkowski content of only one orbit. By their definition, they are related to the first term in the asymptotic expansion of the *area of the ε-neighborhood* of the set. In literature, the function (in ε) of the (inner) area of the ε-neighborhood of a set is sometimes referred to as its *tube function*. For some interesting sets the explicit *tube formulas* for tube functions are given, see e.g. [10] or [11]. We have shown in [21] that the formal class can be recognized from finitely many terms in the asymptotic expansion in ε of the (directed) areas of the ε-neighborhoods of one of its orbits. This question is similar to the famous question: *Can we hear a shape of a drum?*, posed by M. Kac in 1966. It asks if one can reconstruct the equation, that is, the shape of the domain, from *only one* solution. The vibrations of a drum are given by Laplace equation on a domain Ω. By the famous Weyl-Berry conjecture, see [12], the volume of Ω and the box dimension and the Minkowski content of $\partial\Omega$ can be read from the first two terms in the asymptotic expansion of the eigenvalue counting function.

Our object of research in this article are the functions of the (directed) areas of the ε-neighborhoods of orbits of a germ f and their relation to the moduli of analytic classification. To investigate if the analytic class can be seen in functions of ε-neighborhoods of orbits is a natural continuation of the formal classification problem resolved in [21]. The idea is to see to what extent ε-neighborhoods of orbits describe the given germ. This paper was motivated by the question:

Can we read the analytic class of a germ from ε-neighborhoods of its orbits, as functions of parameter $\varepsilon > 0$ and of initial point $z \in \mathbb{C}$?

It is clear that the analytic class, unlike the formal class, cannot be read from any finite jet of parabolic germ, see e.g. [9, Sec. 21]. Accordingly, we are forced to analyse analyticity of the whole functions of the areas of the ε-neighborhoods of orbits, instead of considering only finitely many terms in their expansions. Furthermore, instead of considering only one orbit, we search for a function constant along orbits, that is, well-defined on *space of orbits*. In Section 3, the article gives different results concerning the analyticity properties of the (directed) areas of the ε-neighborhoods of orbits. We reach the conclusion that the *principal parts* in the expansions for f and for the inverse germ $f^{\circ -1}$ (defined in Subsection 1.2) are the only sectorially analytic objects related to the above function. They moreover satisfy a cohomological equation similar to the trivialisation (Abel) equation, standardly used in context of the analytic classification. As a result, the difference of principal parts for f and for $f^{\circ -1}$ is a well-defined function on *space of closed orbits*. We investigate in Sections 2, 4 and 5 if this function reveals the analytic class of a germ. The underlying idea is to characterize the germs with *global principal parts* and to compare this set of germs to the trivial analytic class. The two sets of germs seem unrelated, and in Section 6 we support the observed fact. We study solutions of a special type of cohomological equations, which generalize the equation for principal parts:

$$H \circ f - H = Id^m,\ m \in \mathbb{N}_0.$$

We call them the *m-Abel equations*. Under this notation, the principal parts equation corresponds to 1-Abel equation, and the trivialisation equation to 0-Abel equation. We introduce new classifications of germs imposed by differences of sectorial solutions of m-Abel equations, called *m-conjugacies*. Finally, we show that the 1-classes of germs are *transversal* to the analytic classes of germs. The differences of principal parts on closed orbits are thus not sufficient to read the analytic class of a germ. One question for the future research is the position of higher classes to each other.

Apart from standard use of Abel equation in analytic classification of parabolic germs due to Écalle and Voronin, the other *cohomological equations* in relation to conjugacy problems have been considered for real-line diffeomorphisms by Belitskii, Tkachenko [2] or Lyubich [16]. In [7], Grintchy and Voronin provide analytic classification of resonant complex saddles, based on classification of their t-monodromies, $t \in \mathbb{C}$. They are 2-dimensional *t-shifts*, with two components: the parabolic holonomy map and the return time function. The quadruple obtained as moduli are in relation with the pair of moments corresponding to Abel and higher order cohomological equation for the parabolic germ, which are defined in Section 7. See Subsection 7.2 for more details.

Recall that sectorial analyticity of formal solutions of difference equations (the Stokes phenomenon) appears frequently in problems from nature, see e.g. [20] for insight. The fact that the cohomological equation exhibiting the phenomenon appears in principal parts, due to geometric properties of ε-neighborhoods of orbits (see proof of Proposition 6), is therefore not unexpected.

Finally, the fact that the areas of the ε-neighborhoods of orbits of parabolic germs were exploited successfully in formal problem, [21], as well as the existence of a bond between cohomological equations and classification problems in dynamical systems in general, present the motivation for studying the classification of germs implied by principal parts of ε-neighborhoods of orbits satisfying 1-Abel equation and its possible relation to analytic classification.

In the end, some statements and detailed proofs omitted from the article for the sake of intelligibility can be found in preprints [22] or, in even more detail, in [23].

1.2. Definitions and main results.

Let

$$f(z) = \lambda z + a_1 z^{k+1} + a_2 z^{k+2} + o(z^{k+2}),\ a_i \in \mathbb{C},\ k \in \mathbb{N},$$

$\lambda = \exp(2\pi i m/n),\ m,\ n \in \mathbb{N}$, be a germ of a parabolic diffeomorphism. Without loss of generality, we assume that $\lambda = 1$. Otherwise, instead of f, we consider its appropriate iterate $f^{\circ n}$. Near the origin, the orbits of f form the so-called *Leau-Fatou flower*, see e.g. [15] or [18]. There exist k attracting and k repelling petals – domains accumulating on 0– bisected by equidistant attracting (repelling) directions and tangent to two of them at the origin. The attracting/repelling directions are normalized complex numbers $(-a_1)^{-1/k}$, $a_1^{-1/k}$ respectively. Orbits are tangent to attracting or repelling directions at the origin, see Figure 1.2.

Let V_+ denote any attracting petal of f. Let

$$S^f(z) = \{z_n \mid z_n = f^{\circ n}(z),\ n \in \mathbb{N}\}$$

denote the orbit of f with the initial point z lying in V_+. We recall the asymptotic behavior of z_n from e.g. [18]. For further terms, see [21].

$$z_n = (-ka_1)^{-\frac{1}{k}} \cdot n^{-\frac{1}{k}} + o(n^{-\frac{1}{k}}),\ n \to \infty.$$

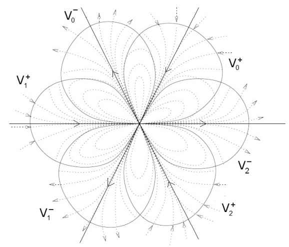

FIGURE 1. Attracting and repelling petals for e.g. $f(z) = z + z^4 + o(z^4)$.

Definition 1. [*see* [21]] *Let* $S^f(z),\ z \in V_+$, *be an attracting orbit of* f, *with initial point* z, *and* $S^f(z)_\varepsilon$ *its* ε*-neighborhood. The* directed area of the ε-neighborhood of the orbit $S^f(z)$ *is the complex number*

$$A^{\mathbb{C}}(z,\varepsilon) = A(S^f(z)_\varepsilon) \cdot t_{S^f(z)_\varepsilon}.$$

Here, $A(S^f(z)_\varepsilon)$ *denotes the area and* $t_{S^f(z)_\varepsilon}$ *the center of the mass of the* ε*-neighborhood.*

Here, for convenience, the directed area is defined in a slightly different manner than in [21]. In [21], the center of mass was replaced by the normalized center of mass, $\frac{t_{S^f(z)_\varepsilon}}{|t_{S^f(z)_\varepsilon}|}$.

Recall the asymptotic expansion of $A^{\mathbb{C}}(z,\varepsilon)$, as $\varepsilon \to 0$, from [21]. It follows directly by adding expansions from Lemmas 4 and 5 in [21]:

$$\begin{aligned} A^{\mathbb{C}}(z,\varepsilon) &= q_1\varepsilon^{1+\frac{2}{k+1}} + q_2\varepsilon^{1+\frac{3}{k+1}} + \ldots + q_{k-1}\varepsilon^{1+\frac{k}{k+1}} + q_k\varepsilon^2\log\varepsilon + \\ &\quad + H^{f,V_+}(z)\varepsilon^2 + q_{k+1}\varepsilon^{2+\frac{1}{k+1}}\log\varepsilon + R(z,\varepsilon),\ R(z,\varepsilon) = O(\varepsilon^{2+\frac{1}{k+1}}), \\ &\qquad k\in\mathbb{N},\ k>1,\ q_i\in\mathbb{C},\ i=1,\ldots k+1. \end{aligned} \tag{2}$$

The above expansion and formulas for the coefficients given in [21] hold *in the case* $k > 1$. In the *special case* $k = 1$, we have the expansion:

$$\begin{aligned} A^{\mathbb{C}}(z,\varepsilon) &= q_1\varepsilon^2\log\varepsilon + H^{f,V_+}(z)\varepsilon^2 + \\ &\quad + q_2\varepsilon^{\frac{5}{2}}\log\varepsilon + R(z,\varepsilon),\ R(z,\varepsilon) = O(\varepsilon^{\frac{5}{2}}),\ q_1,\ q_2\in\mathbb{C}. \end{aligned} \tag{3}$$

The coefficients are given by slightly different formulas than in [21] (divergent for $k = 1$), but the properties of the expansions are the same. For more details, see Lemma 1 in the Appendix. In above expansions, q_i, $i = 1,\ldots,k+1$, are complex functions of k and of the first i coefficients of f. They *do not depend on the initial point.* The coefficient $H^{f,V_+}(z)$ is the first coefficient that depends on the initial point z. It is a well-defined function in z on V_+.

Definition 2. The principal, initial point dependent, part of the directed area of the ε-neighborhoods of orbits in V_+ *is the first coefficient* $H^{f,V_+}(z)$ *in the expansion* (2) *depending on the initial point* z*, regarded as a function of* $z \in V_+$*,* $z \mapsto H^{f,V_+}(z)$.

For simplicity, we call function $z \mapsto H^{f,V_+}(z)$ only *the principal part of area for* f *on* V_+. Naturally, on a repelling sector V_-, we define *the principal part of area for* $f^{\circ -1}$ *on* V_- as the first coefficient that depends on the initial point in the expansion (2) for the orbit $S^{f^{\circ -1}}(z)$ of the inverse germ $f^{\circ -1}$, regarded as function $z \mapsto H^{f^{\circ -1},V_-}(z)$, $z \in V_-$.

Let us comment shortly on properties of $A^{\mathbb{C}}(z,\varepsilon)$, as function of $\varepsilon > 0$ and of $z \in V_+$. They justify why we concentrate on the principal parts as the only parts that display analytic property. The results are detailed and proven in Section 3. We show that, for fixed initial point $z \in V_+$, the remainder $R(\varepsilon, z)$ in (2) cannot be fully expanded in a power-logarithmic scale in ε, as $\varepsilon \to 0$. It has accumulation of singularities at $\varepsilon = 0$. Furthermore, for $\varepsilon > 0$ fixed, $A^{\mathbb{C}}(z,\varepsilon)$ is not analytic in z on any sector around attracting direction. However, we prove Theorem 1 below about sectorial analyticity of principal parts.

For simplicity, we consider only the germs from the model formal class $(k = 1, \rho = 0)$, that is, formally equivalent to the *model* f_0,

$$f_0(z) = Exp\big(z^2 \frac{d}{dz}\big) = \frac{z}{1-z}. \tag{4}$$

Furthermore, we assume f is *prenormalized*. The first normalizing change of variables being already made, we admit only changes of variables tangent to the identity. All such diffeomorphisms are of the form:

$$f(z) = z + z^2 + z^3 + o(z^3).$$

There exists only one attracting petal V_+ (around negative real axis) and only one repelling petal V_- (around positive real axis). We denote the functions H^{f,V_+} and $H^{f^{\circ -1},V_-}$ simply by H^f and $H^{f^{\circ -1}}$. The assumption $\rho = 0$ is not essential; the same analysis could be performed in any formal class $(k = 1, \rho)$ (Theorems 1, 3, 4 hold, in Theorem 2 a slight difference in h_{α_0,α_1} when the right-hand side contains a constant term). However, it is difficult to compute explicit examples (Example 3, 4) for model germs in other formal classes.

Theorem 1 (Properties of principal parts of areas for a germ)**.** *The principal parts* H^f *and* $H^{f^{\circ -1}}$ *are analytic functions on petals* V_+ *and* V_- *respectively. Moreover,* H^f *and* $H^{f^{\circ -1}}$ *are, up to explicit constants, related to the unique sectorially analytic solutions without constant* *term, H_+ on V_+ and H_- on V_-, of difference equation:*

$$H(f(z)) - H(z) = -\pi z, \tag{5}$$

The following explicit formulas hold:

$$H_+(z) - \frac{\pi}{4} + i\pi^2 = H^f(z), \quad z \in V_+,$$
$$H_-(z) - \frac{\pi}{4} = \pi z - H^{f^{\circ -1}}(z), \quad z \in V_-.$$

The equation (5) resembles to the trivialization (Abel) equation for a parabolic germ:

$$\Psi(f(z)) - \Psi(z) = 1 \tag{6}$$

There exist analytic solutions on petals, Ψ_+ on V_+ and Ψ_- on V_-, the so-called *Fatou coordinates*. Comparing them on intersections of petals one obtains *Ecalle-Voronin moduli of analytic classification*, see e.g. [6], [27] or [5], which give the analytic class of f. See Subsection 6.1. The Fatou coordinates Ψ_+ and Ψ_- glue to a global Fatou coordinate, analytic in some punctured neighborhood of the origin, if and only if f belongs to the model analytic class (class of f_0 from (4)). That is, if and only if f is a time-one map of a holomorphic vector field.

We recall here the definition of *cohomological difference equations*, which generalize both equations (5) and (6). The name *cohomological* stems from algebraic-topological approach to dynamical systems. The *coboundary operator* δ for a dynamical system generated by f, acting on a homomorphism H, is given by $\delta H(f, z) = H(f(z)) - H(z)$. For details, see e.g. [13]. The notion of cohomological equations is known in literature, see e.g. [2], [9], [16] or [14, Section A.6].

Definition 3 (A cohomological equation)**.** A cohomological equation for a germ f with the right-hand side $g \in \mathbb{C}\{z\}$, $g \not\equiv 0$, *is the equation*

$$H(f(z)) - H(z) = g(z). \tag{7}$$

The function H that satisfies (7) *on some domain is called* a solution *of the cohomological equation on the given domain. In particular, if*

$$g = C \cdot Id^m,\ C \in \mathbb{C},\ m \in \mathbb{N}_0,$$

we call equation (7) *the m-*Abel equation*.*

The Abel equation (6) is thus 0-Abel and our principal part equation (5) the 1-Abel equation for f.

In Section 2, we discuss solvability of cohomological equations. The results on existence of sectorially analytic solutions are adapted from [14] or [9, Proof of the Theorem 21.5]. Our result is Theorem 2, that gives necessary and sufficient conditions on a germ f in terms of right-hand side, for the cohomological equation to have a globally analytic solution. The proof is in Section 2.

Let the right-hand side g of (7) be *of multiplicity* l. That is,

$$g(z) = \alpha_l z^l + o(z^l) \in \mathbb{C}\{z\},\ \alpha_l \neq 0,\ l \in \mathbb{N}_0. \tag{8}$$

In the cases $l = 0$ ($\alpha_0 \neq 0$, i.e., the multiplicity of g is 0) or $l = 1$ ($\alpha_0 = 0,\ \alpha_1 \neq 0$, i.e., the multiplicity of g is 1), we define:

$$h_{\alpha_0,\alpha_1}(z) = -\frac{\alpha_0}{z} + \alpha_1 \log z.$$

Theorem 2 (Globally analytic solution of a cohomological equation)**.** *Let f be a parabolic germ from the model formal class. Let $g \in \mathbb{C}\{z\}$, $g \not\equiv 0$, be of multiplicity $l \in \mathbb{N}_0$. The cohomological equation*

$$H(f(z)) - H(z) = g(z)$$

has a (unique up to a constant) globally analytic solution on some neighborhood of origin if and only if the germ f is of the form[1]

$$f(z) = \begin{cases} \varphi^{-1}\Big(h^{-1}_{\alpha_0,\alpha_1}\Big(h_{\alpha_0,\alpha_1}\big(\varphi(z)\big) + g(z)\Big)\Big) & ,\ l = 0,\ 1, \\ \varphi^{-1}\left(\varphi(z)\cdot\left(1 + \frac{l-1}{\alpha_l}\frac{g(z)}{\varphi(z)^{l-1}}\right)^{\frac{1}{l-1}}\right) & ,\ l \in \mathbb{N},\ l \geq 2, \end{cases} \tag{9}$$

for some $\varphi \in z + z^2\mathbb{C}\{z\}$. The globally analytic solution H is given by

$$H(z) = \begin{cases} h_{\alpha_0,\alpha_1} \circ \varphi(z) & ,\ l = 0,\ 1, \\ \frac{\alpha_l}{l-1}\varphi(z)^{l-1} & ,\ l \in \mathbb{N},\ l \geq 2. \end{cases} \tag{10}$$

We use the term *globally analytic* in a slightly incorrect manner. In the case when g contains a constant term, H contains the term $-1/z$ in the asymptotic expansion. When the linear term of g is non-zero, H contains a logarithmic term. The *global analyticity* of solution H of (7) means the global analyticity of solution R, $H(z) = -\frac{\alpha_0}{z} + \alpha_1 \log z + R(z)$, of the modified equation

$$R(f(z)) - R(z) = g(z) + \alpha_0\left(\frac{1}{f(z)} - \frac{1}{z}\right) - \alpha_1 \log\big(\frac{f(z)}{z}\big).$$

Based on Theorem 2, in Theorem 3 we characterize the germs whose principal parts are *globally analytic*. The theorem shows that global principal parts are not the *rule*. Its proof is in Section 4.

[1]If $\alpha_1 \neq 0$, then h_{α_0,α_1} contains a logarithmic term and is not a well-defined nor invertible function on some neighborhood of origin. We work with two invertible branches on overlapping sectors, and then glue the sectors together in function f analytic at zero by Riemann's theorem on removable singularities.

Theorem 3 (*Global* principal parts of areas)**.** *The principal parts* $(H^f - i\pi^2)$ *on* V_+ *and* $(\pi \cdot Id - H^{f^{\circ -1}})$ *on* V_- *glue to a global analytic function on a neighborhood of origin if and only if*

$$f(z) = \varphi^{-1}\left(e^z \cdot \varphi(z)\right),$$

for some $\varphi \in z + z^2\mathbb{C}\{z\}$ *analytic. The principal parts are given by*

$$H^f(z) = -\pi \log \varphi(z) + i\pi^2 - \frac{\pi}{4}, \quad z \in V_+,$$
$$H^{f^{\circ -1}}(z) = \pi z + \pi \log \varphi(z) + \frac{\pi}{4}, \quad z \in V_-.$$

The branches of complex logarithm are determined by the petals.

In Section 5, using Theorem 3, we provide examples that suggest that the set of germs with global principal parts is not related to the model analytic class. Even the principal parts of the model f_0 are not global. This motivates us to introduce new classifications of germs with respect to m-Abel equations, $m \in \mathbb{N}_0$, in Section 6. This puts the Abel equation (6) and the principal parts equation (5) in a more general context. The new *m-conjugacy classes* are described by pairs of analytic germs that we call *m-moments*. They represent the differences of sectorial solutions of m-Abel equations as functions defined on a quotient space of closed orbits. Their construction mimics the Écalle-Voronin moduli from Subsection 6.1 on higher equations. The analytic classes correspond to 0-moments. Similar construction can be found in [7, Section 4.2] in different context, as components of moduli of classification of 2-dimensional shifts, without mention of cohomological equations or higher moments.

Theorem 4 (*Transversality* theorem)**.** *Let* Φ *be a mapping that associates to each germ* f *its* 1*-moment. The mapping* Φ *restricted to any analytic class is* surjective *onto the set of all* 1*-moments.*

The precise formulation and the proof can be found in Subsection 6.3, where the question of injectivity is also shortly addressed.

By Theorem 4, each 1-class admits a representative in any analytic class. In particular, there exist germs in each analytic class with global principal parts. This gives the *negative* answer to our question of reading the analytic class from principal parts of areas. However, it opens new questions of the geometric meaning of new classifications of germs and of the relative position of higher conjugacy classes to each-other.

2. Solutions of cohomological equations

Here we adapt the statements from [14, Section A.6] dealing with the right-hand side $g(z) = O(z^2)$ to the more general cases $g \in \mathbb{C}\{z\}$.

The argument is standard, and the same is used, for example, in the proof of the sectorial normalization theorem for parabolic germs that can be found in e.g. [9, p. 378] or [15, p. 48]. We only state important steps of the proof.

Proposition 1 (Formal and sectorial solutions of cohomological equations)**.** *Let* $g \in \mathbb{C}\{z\}$, $g(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + o(z^2)$, $\alpha_i \in \mathbb{C}$, $i \in \mathbb{N}_0$. *There exists a unique* formal series solution $\widehat{H}$ *of equation* (7) *without constant term of the form*

$$\widehat{H}(z) \in -\frac{\alpha_0}{z} + \alpha_1 \log z + z\mathbb{C}[[z]]. \tag{11}$$

All other formal solutions in the given scale are obtained by adding an arbitrary constant term. Furthermore, there exist unique (without constant term) sectorially analytic solutions*,* H_+ *on* V_+ *and* H_- *on* V_-*, with asymptotic expansion* (11)*, as* $z \to 0$*.*

Proof. The proof of existence and uniqueness of the formal solution is straightforward, solving the cohomological equation (7) term by term. To prove existence of sectorially analytic solutions, instead of H, we consider R,

$$R(z) = H(z) + \frac{\alpha_0}{z} - \alpha_1 \log z.$$

By (7), R satisfies the difference equation

$$R(f(z)) - R(z) = \delta(z), \tag{12}$$

where $\delta \in z^2\mathbb{C}\{z\}$. Now we directly apply results from [14, A.6] or [9, Proof of the Theorem 21.5, p. 378] and find two sectorially analytic functions on petals, R_+ on V_+ and R_- on V_-, that satisfy equation (12). Moreover, they admit $\widehat{R}(z) = \widehat{H}(z) + \frac{\alpha_0}{z} - \alpha_1 \log z \in z\mathbb{C}[[z]]$ as their asymptotic expansion, as $z \to 0$. Let us describe shortly the standard idea of solving cohomological equations. We consider the following series:

$$-\sum_{n\geq 0} \delta\big(f^{\circ n}(z)\big),\ z \in V_+, \tag{13}$$

$$\sum_{n\geq 1} \delta\big(f^{\circ(-n)}(z)\big),\ z \in V_-. \tag{14}$$

It can be proven that the above series converge uniformly on compact subsets of V_+, V_- respectively. By Weierstrass theorem, they converge to analytic functions on petals, which we denote by R_+ and R_-:

(15)

$$R_+(z) = -\sum_{n\geq 0} \delta\big(f^{\circ n}(z)\big),\ z \in V_+, \quad R_-(z) = \sum_{n\geq 1} \delta\big(f^{\circ(-n)}(z)\big),\ z \in V_-.$$

It can be furthermore checked that both R_+ and R_- admit $\widehat{R}$ as their asymptotic expansion on petal, as $z \to 0$.

The uniqueness of sectorial analytic solutions R_+ and R_- with asymptotic expansion $\widehat{R}$ is easy to prove. On V_+, iterating equation (12) along the orbit of f, summing the iterations and passing to the limit, we see that any analytic solution of the type $O(z)$ of (12) is necessarily given by the same convergent series (13), and is thus unique. The same can be concluded for V_- and formula (14).

Finally, the solutions of initial equation (7) are given by

$$H_\pm(z) = R_\pm(z) - \frac{\alpha_0}{z} + \alpha_1 \log z \text{ on } V_\pm,$$

where $R_\pm$ are as in (15). On each petal, we choose the appropriate branch of logarithm. □

In the proof of Theorem 2 from Section 1, we need Proposition 2 below. It can be proven easily. Note that the assumptions on the existence of formal expansions are crucial for the implication to hold.

Proposition 2. *Let $\widehat{g}$, $\widehat{T} \in \mathbb{C}[[z]]$, $h \in \mathbb{C}\{z\}$ non-constant, such that*

$$\widehat{T} = h \circ \widehat{g}.$$

Then $\widehat{T}$ is analytic if and only if $\widehat{g}$ is analytic.

Proof of Theorem 2. We consider two cases separately.

i) $\mathbf{l \geq 2}$. The formal solution $\widehat{H} \in z\mathbb{C}[[z]]$ is of the form

$$\widehat{H}(z) = \frac{\alpha_l}{l-1} z^{l-1} + o(z^{l-1}).$$

Equivalently, we can write

$$\widehat{H}(z) = \frac{\alpha_l}{l-1} \widehat{\varphi}(z)^{l-1},$$

where $\widehat{\varphi} \in z + z^2\mathbb{C}[[z]]$. By Proposition 2, H is globally analytic if and only if φ is globally analytic.

Suppose now that H is globally analytic. Putting $H = \frac{\alpha_l}{l-1}\varphi^{l-1}$ in equation (7), we can uniquely express f:

$$f(z) = \varphi^{-1}\left(\left(\varphi(z)^{l-1} + \frac{l-1}{\alpha_l} g(z)\right)^{\frac{1}{l-1}}\right). \tag{16}$$

Here, $\varphi(z)^{l-1} \sim z^{l-1}$ and $g(z) \sim \alpha_l z^l$, as $z \to 0$. The $(l-1)$-th root we take is uniquely determined, since f and φ are tangent to the identity. Formula (16) easily transforms to (9).

Conversely, if f is of the form (9) for $\varphi \in z + z^2\mathbb{C}\{z\}$, it is easy to see that $H = \frac{\alpha_l}{l-1}\varphi^{l-1}$ satisfies equation (7) for f and that the formal expansion is of the form (11). By uniqueness in Proposition 1, H is the unique analytic solution of (7).

ii) $\mathbf{l}=\mathbf{0},\mathbf{1}$. Similarly, the formal solution $\widehat{H}$ can be written as

$$\widehat{H}(z)=h_{\alpha_0,\alpha_1}\circ\widehat{\varphi}(z),$$

where $\widehat{\varphi}\in z+z^2\mathbb{C}[[z]]$. Equivalently,

$$\widehat{H}(z)=-\frac{\alpha_0}{z}+\alpha_1\log z+r\left(\frac{\widehat{\varphi}(z)-z}{z}\right),$$

where r is a nonconstant analytic germ. Now, by Proposition 2, $\widehat{H}$ is *globally analytic* (in the sense that $\widehat{H}(z)+\frac{\alpha_0}{z}-\alpha_1\log z$ is globally analytic) if and only if $\widehat{\varphi}$ is. We proceed as in *i*). The function h_{α_0,α_1} in expression (9) can be regarded as function with two branches. In the case $\alpha_0\neq 0$, it is the Fatou coordinate for the vector field $X_{1,\lambda}$, $\lambda=2\pi i\frac{\alpha_1}{\alpha_0}$, see e.g.[15]. In the case $\alpha_0=0$, it is merely a logarithmic function. It is then invertible on sectors. □

3. Analytic properties of $A^{\mathbb{C}}(z,\varepsilon)$ in $\varepsilon>0$ and in $z\in V_+$

The missing proofs of propositions can be found in the Appendix.

3.1. Properties of $\varepsilon\mapsto A^{\mathbb{C}}(z,\varepsilon)$, $\varepsilon>0$.

Proposition 3 below presents an obstacle for extending function $\varepsilon\mapsto A^{\mathbb{C}}(z,\varepsilon)$ from real positive to complex ε, by means of formal series.

Proposition 3 (Nonexistence of full power-logarithmic asymptotic expansion in ε)**.** *Let $z\in V_+$ be fixed. After a certain number of terms, an asymptotic expansion of $A^{\mathbb{C}}(z,\varepsilon)$ in a power-logarithmic scale, as $\varepsilon\to 0$, does not exist. There exists $l\in\mathbb{N}$, such that the remainder $R(z,\varepsilon)$ in (2) is of the form:*

$$R(z,\varepsilon)=h_1(z)g_1(\varepsilon)+\ldots+h_{l-1}(z)g_{l-1}(\varepsilon)+h(z,\varepsilon),$$
$$h(z,\varepsilon)=O\big(g_l(\varepsilon)\big),\ \varepsilon\to 0.$$

The monomials $g_i(\varepsilon)$ are of power-logarithmic type in ε, of increasing flatness at zero, but the limit

$$\lim_{\varepsilon\to 0}\frac{h(z,\varepsilon)}{g_l(\varepsilon)}$$

does not exist.

Proposition 4 expresses an obstacle for the analytic continuation of $A^{\mathbb{C}}(z,\varepsilon)$ in ε on the neighborhood of the positive real line.

Let us denote by $z_n = f^{\circ n}(z)$, $n \in \mathbb{N}_0$, the points of the orbit. Let $d_n = |z_n - z_{n+1}|$, $n \in \mathbb{N}_0$, denote the distances between consecutive points of the orbit and let

$$\varepsilon_n = \frac{d_n}{2},\ n \in \mathbb{N}_0.$$

Note that $\varepsilon_n \to 0$, as $n \to \infty$.

The loss of regularity of $\varepsilon \mapsto A^{\mathbb{C}}(z,\varepsilon)$ at points ε_n at which separation of the tail and the nucleus occurs is related to the different rate of growth of the tail and the nucleus of ε-neighborhoods in ε, due to their different geometry (overlapping discs in nucleus, disjoint discs in tail).

Proposition 4 (Accumulation of singularities at $\varepsilon = 0$)**.** *Let $\varepsilon_0 > 0$. The function $\varepsilon \mapsto A^{\mathbb{C}}(z,\varepsilon)$ is of class C^1 on $(0,\varepsilon_0)$ and C^∞ on open subintervals $(\varepsilon_{n+1},\varepsilon_n)$, $n \in \mathbb{N}_0$. However, in ε_n, $n \in \mathbb{N}_0$, the second derivative is unbounded from the right:*

$$\lim_{\varepsilon\to\varepsilon_n-} \frac{d^2}{d\varepsilon^2} A^{\mathbb{C}}(z,\varepsilon) \in \mathbb{C},\ \lim_{\varepsilon\to\varepsilon_n+} \left|\frac{d^2}{d\varepsilon^2} A^{\mathbb{C}}(z,\varepsilon)\right| = +\infty.$$

3.2. **Properties of $z \mapsto A^{\mathbb{C}}(z,\varepsilon)$, $z \in V_+$.**

Let us fix $\varepsilon > 0$. Proposition 5 states that function $z \mapsto A^{\mathbb{C}}(z,\varepsilon)$ is not analytic on the attracting (repelling) petal.

Let $S^{\pm}(\varphi, r)$, $\varphi \in (0,\pi)$, $r > 0$, denote open sectors of opening 2φ and radius $r > 0$, bisected by the attracting (repelling) direction.

Proposition 5. *Let $\varepsilon > 0$. The function $z \mapsto A^{\mathbb{C}}(z,\varepsilon)$ is not analytic on any open sector $S^+(\varphi, r)$. The function $z \mapsto A^{\mathbb{C},f^{\circ-1}}(z,\varepsilon)$ is not analytic on any open sector $S^-(\varphi, r)$.*

3.3. **Properties of the principal parts of areas.**

Having described *bad* properties of the directed areas of orbits, we concentrate only on their principal parts:

$$z \mapsto H^f(z),\ z \in V_+, \quad \text{and } z \mapsto H^{f^{\circ-1}}(z),\ z \in V_-.$$

We prove Theorem 1 about analyticity on petals. The relation of principal parts with cohomological equation (5) in theorem was inspired by Proposition 6, which follows from the geometry of ε-neighborhoods.

Proposition 6. *The principal parts of areas H^f and $H^{f^{\circ-1}}$ satisfy the following difference equations:*

$$H^f(f(z)) - H^f(z) = -\pi z,\ z \in V_+, \tag{17}$$

$$H^{f^{\circ-1}}(f^{\circ-1}(z)) - H^{f^{\circ-1}}(z) = -\pi z,\ z \in V_-. \tag{18}$$

Here, V_+ denotes any attracting and V_- any repelling petal.

Proof. Let us first derive (17) for H^f on V_+. By the definition of the directed area, we have that, for $\varepsilon < \varepsilon_z$ small enough with respect to z,

$$A^{\mathbb{C}}(z,\varepsilon) = A^{\mathbb{C}}(f(z),\varepsilon) + z\cdot\varepsilon^2\pi,\ z\in V_+. \tag{19}$$

Putting the expansion (2) in (19), we get that

$$\big[H^f(z) - H^f(f(z))\big]\varepsilon^2 + \big(R(z,\varepsilon) - R(f(z),\varepsilon)\big) = \varepsilon^2\pi.$$

By (2), $R(z,\varepsilon) - R(f(z),\varepsilon) = o(\varepsilon^{2+\frac{1}{k+1}})$. Dividing by ε^2 and passing to the limit as $\varepsilon \to 0$, (17) follows. Equation (18) is derived analogously, considering directed areas of orbits of $f^{\circ -1}$ on a repelling petal. □

Proof of Theorem 1.

We analyse the coefficient $H^f(z)$ in front of ε^2 in expansion (3), as function of $z \in V_+$. Let us remind, the tail is the part of the ε-neighborhood which is the union of disjoint ε-discs, while the nucleus is the remaining part with overlapping discs. We denote by $z \mapsto H_N^f(z),\ z \mapsto H_T^f(z),\ z\in V_+$, the principal parts of the directed area of the nucleus and of the tail respectively. We have:

$$H^f(z) = H_N^f(z) + H_T^f(z),\ z\in V_+. \tag{20}$$

By Lemma 1(*iii*) in the Appendix, the principal part for the nucleus is constant and equal to

$$H_N^f(z) = -\frac{\pi}{4}(1+\log 4),\ z\in V_+. \tag{21}$$

The dependence on z of the principal part comes from the tail. The directed area of the tail of $S^f(z)_\varepsilon$ is by its definition equal to:

$$\big(A(T_\varepsilon)\cdot t_{T_\varepsilon}\big)(z) = \varepsilon^2\pi\cdot\sum_{l=0}^{n_\varepsilon} f^{\circ l}(z). \tag{22}$$

Here, n_ε is the index of separation of the tail and the nucleus. Obviously, $n_\varepsilon \to \infty$, as $\varepsilon \to 0$. Using the formal expansion of $f^{\circ l}(z)$, as $l \to \infty$, from Lemma 1(*i*) in the Appendix,

$$f^{\circ l}(z) = -l^{-1} + q(z)l^{-2} + o(l^{-2}), \tag{23}$$

and by integral approximation of the sum, we obtain

$$\sum_{l=0}^{n} f^{\circ l}(z) = -\log n + C(z) + o(1),\ n\to\infty,\ z\in V_+. \tag{24}$$

Here, $C(z)$ denotes the constant term in the expansion of the sum $\sum_{l=0}^{n} f^{\circ l}(z)$, as $n\to\infty$. Putting (24) in (22), we get:

$$\big(A(T_\varepsilon)\cdot t_{T_\varepsilon}\big)(z) = \varepsilon^2\pi\cdot\big(-\log n_\varepsilon + C(z) + o(1)\big),\ \varepsilon\to 0. \tag{25}$$

Further, putting the expansion for n_ε from Lemma 1.(ii) in the Appendix in (25), we conclude that

$$H_T^f(z) = \frac{\pi}{2}\log 2 + \pi \cdot C(z),\ z \in V_+. \tag{26}$$

By (20), (21) and (26), we get:

$$H^f(z) = -\frac{\pi}{4} + \pi \cdot C(z),\ z \in V_+. \tag{27}$$

Our next step is to prove analyticity of the function H^f given by (27) on V_+. To this end, we consider the unique analytic solution on V_+ without constant term of equation (5):

$$H(f(z)) - H(z) = -\pi z. \tag{28}$$

By the proof of Proposition 1, it is given by the limit

$$H_+(z) = \pi \lim_{n\to\infty}\left(\sum_{l=0}^{n} f^{\circ l}(z) - \log f^{\circ(n+1)}(z)\right), \tag{29}$$

which was proven to converge pointwise to an analytic function on V_+.

To prove analyticity of H^f on V_+, it suffices to show that the expression (27) for $H^f(z)$ coincides pointwise with $H_+(z)$ in (29), up to a constant. For a fixed z, by (24), we estimate the first terms in the asymptotic expansion of $\sum_{l=0}^{n} f^{\circ l}(z) - \log f^{\circ(n+1)}(z)$, as $n \to \infty$:

$$\begin{aligned}&\sum_{l=0}^{n} f^{\circ l}(z) - \log f^{\circ(n+1)}(z) = \\ &\quad = -\log n + C(z) + o(1) - \log f^{\circ(n+1)}(z) = C(z) - i\pi + o(1).\end{aligned} \tag{30}$$

Here, $C(z)$ is as defined above. The last equality follows using (23):

$$\begin{aligned}&-\log^+ f^{\circ(n+1)}(z) - \log^- n = -\log^+\left(-\frac{1}{n}\big(1+o(1)\big)\right) - \log^- n = \\ &\quad = -\log^- n + \log^+(-n) - \log^-(1+o(1)) = -i\pi + o(1),\ n \to \infty.\end{aligned}$$

Here, $\log^- z$ is the principal branch of logarithm, $\arg z \in (-\pi, \pi)$, and $\log^+ z$ the branch for $\arg z \in (0, 2\pi)$.

Passing to the limit in (29), by (30), we get the pointwise equality:

$$H_+(z) = \pi \cdot C(z) - i\pi^2,\ z \in V_+.$$

By (27), we conclude

$$H^f(z) + \frac{\pi}{4} - i\pi^2 = H_+(z).$$

Therefore, since H_+ is analytic on V_+, H^f is also analytic on V_+.

Analyticity of $H^{f^{\circ -1}}$ on V_- can be proven in the same manner, considering inverse diffeomorphism $f^{\circ -1}$, and comparing $H^{f^{\circ -1}}$ with sectorial solution H_- of equation (28) on V_-. □

4. Applications of Theorem 2

4.1. ***Trivial*** application: global solution of the Abel equation.

The trivialization (Abel) equation for a parabolic germ f:

$$\Psi(f(z)) - \Psi(z) = 1. \tag{31}$$

We use Theorem 2 to derive a well-known result by Écalle and Voronin: a germ f is analytically conjugated to the model f_0 if and only if equation (31) has a global solution Ψ. Of course this is not new, and we put it here only as a trivial example.

Proof by Theorem 2. Abel equation is a cohomological equation with right-hand side $g \equiv 1$. Therefore, $h_{1,0}(z) = -1/z$. By (9), there exists a global analytic solution of (31) if and only if f is given by

$$f = \varphi^{-1}\left(-\frac{1}{-\frac{1}{\varphi}+1}\right) = \varphi^{-1} \circ f_0 \circ \varphi,$$

for some $\varphi \in z + z^2\mathbb{C}\{z\}$. φ realizes the conjugacy between f and f_0. The solution is unique up to additive constant and, by (10), of the form $\Psi = \Psi_0 \circ \varphi,\ \Psi_0(z) = -1/z$. □

4.2. ***Nontrivial*** application: Theorem 3.

Proof of Theorem 3. The theorem is a direct consequence of Theorems 1 and 2. By Theorem 1, the principal parts are explicitely related to the sectorial solutions of the cohomological equation with right-hand side $g = -\pi \cdot \mathrm{Id}$. By Theorem 2, this equation has a global solution if and only if $f(z) = \varphi^{-1}(\varphi(z) \cdot e^z)$, for some $\varphi \in z + z^2\mathbb{C}\{z\}$. □

We provide here two simple examples of parabolic germs with *global principal parts of areas* from Theorem 3.

Example 1 (Germs with *global* principal parts)**.**

(1) $f(z) = z \cdot e^z$, *for* $\varphi = Id$,
(2) $f(z) = -\log(2 - e^z)$, *for* $\varphi(z) = 1 - e^{-z}$.

5. Counterexamples: trivial analytic class versus set of germs with global principal parts

We consider germs belonging to the formal class $(k = 1, \rho = 0)$, prenormalized. This standard restriction allows simpler presentation of analytic classification. Changes of variables are tangent to the identity.

Recall that the analytic class of a germ is *hidden* in differences of sectorial Fatou coordinates, or sectorial solutions of Abel equation, on intersections of petals. We give examples that illustrate the *inability of reconstructing the analytic class* by subtracting principal parts (or sectorial solutions of 1-Abel equation) on intersections of petals.

We denote the upper and the lower component of $V_+ \cap V_-$ by:

$$V^{up} = \{z \in V_+ \cap V_- \mid Im(z) > 0\},\ V^{low} = \{z \in V_+ \cap V_- \mid Im(z) < 0\}.$$

By Riemann's theorem on removable singularities, the trivial differences of sectorial solutions of (5) on intersections of petals (up to constant $2\pi i$ from different branches of logarithm):

$$H_+(z) - H_-(z) \equiv 0,\ z \in V^{up};\ \ H_-(z) - H_+(z) \equiv -2\pi i,\ z \in V^{low}, \tag{32}$$

corresponds to the fact that (5) has a global analytic solution.

Let $\mathcal{C}_0$ be the class of diffeomorphisms analytically conjugated to f_0. Denote by $\mathcal{S}$ the set of germs such that (5) has a global analytic solution. By Theorem 3, it follows that

$$\mathcal{S} = \Big\{ f(z) = z + z^2 + z^3 + o(z^3) \,\Big|\, f = \varphi^{-1}(e^z \cdot \varphi(z)),\ \varphi(z) = z + z^2\mathbb{C}\{z\} \Big\}.$$

Example 2 shows that $\mathcal{S} \cap \mathcal{C}_0$ is nonempty. Furthermore, none of the sets is a subset of the other. The trivial analytic class and the *trivial* class with respect to 1-Abel equation are in general position. To conclude, information given by differences $H_+ - H_-$ on $V^{up} \cup V^{low}$ is insufficient for determining the analytic class.

Example 2.

$$\begin{aligned}
&(i)\ f(z) = -\log(2 - e^z) \in \mathcal{S} \cap \mathcal{C}_0,\\
&(ii)\ g(z) = ze^z,\ g(z) \in \mathcal{S},\ g(z) \notin \mathcal{C}_0,\\
&(iii)\ f_0(z) \in \mathcal{C}_0,\ f_0(z) \notin \mathcal{S}.
\end{aligned}$$

In (i)*, we take* $\varphi^{-1}(z) = -\log(1 - z)$ *for both classes. Example* (ii) *follows from the fact that no entire function is analytically conjugated to* f_0*, see* [1]*. Example* (iii) *follows from Example 3 below.*

In Example 3, we compute the differences $H_+ - H_-$ for the model germ (with global Fatou coordinate), and get a nontrivial cocycle. We apply the Borel-Laplace technique directly to equation (5). The method is standard, a similar one to be found in e.g.[5, Example 2],[4]. The example is explained in details in preprint [23, Ex.3.5].

Example 3 (The differences for $f_0(z)=\frac{z}{1-z}$)**.** *We substitute* $\widehat{H}(z)=-\pi\log z+\pi\widehat{R}(z)$, $\widehat{R}\in z\mathbb{C}[[z]]$, *in equation* (5) *for* f_0*. We get:*

$$\widehat{R}(f_0(z))-\widehat{R}(z)=-z+\log\frac{f_0(z)}{z}.$$

By the change of variables $w=-1/z$, $\widehat{\widehat{R}}(w)=\widehat{R}\circ\chi$, $\chi(w)=-1/w$,

$$\widehat{\widehat{R}}(w+1)-\widehat{\widehat{R}}(w)=w^{-1}-Log(1+w^{-1})=\sum_{k=2}^{\infty}(-1)^k\frac{w^{-k}}{k}. \tag{33}$$

The right-hand side of this equation is of the type $w^{-2}\mathbb{C}\{w^{-1}\}$*. Put*

$$b(w)=\sum_{k=2}^{\infty}\frac{(-1)^k}{k}w^{-k}.$$

Applying the Borel transform to (33)*, we get*

$$\mathcal{B}\widehat{\widehat{R}}(\xi)=\frac{\mathcal{B}b(\xi)}{e^{-\xi}-1},\quad \mathcal{B}b(\xi)=\frac{e^{-\xi}+\xi-1}{\xi}.$$

It can be shown that function $\xi\mapsto\mathcal{B}\widehat{\widehat{R}}(\xi)$ *has 1-poles at* $2i\pi\mathbb{Z}^*$ *in directions* $\pm i$*, and is exponentially bounded and analytic in every other direction. For details, see* [5]*. Therefore, Laplace transform yields two analytic solutions,* $\widetilde{R}^+$ *on* $W_+=\{w\,|\,Re(we^{i\theta})>\beta_0,\ \theta\in(-\pi/2,\pi/2)\}$, *and* $\widetilde{R}^-$ *on* $W_-=\{w\,|\,Re(we^{i\theta})>\beta_0,\ \theta\in(\pi/2,3\pi/2)\}$, *where* $\beta_0>0$*. By the residue theorem, for* $w\in W^{up}=\{w\,|\,Im(w)>\beta_0\}$*, we have:*

$$\begin{aligned}\widetilde{R}^+(w)-\widetilde{R}^-(w)&=\int_0^{\infty\cdot e^{i\theta_1}}\frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi}-1}d\xi-\int_0^{\infty\cdot e^{i\theta_2}}\frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi}-1}d\xi=\\&=-2\pi i\cdot\sum_{k=1}^{\infty}Res\Big(\frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi}-1},\xi=-2\pi ik\Big)=2\pi i\frac{e^{2\pi i\cdot w}}{1-e^{2\pi i\cdot w}}.\end{aligned}$$

Here, $\theta_1\in(-\pi/2,\pi/2)$ *and* $\theta_2\in(\pi/2,3\pi/2)$ *are close to* $-\pi/2$*.*

Similarly, for $w\in W^{low}=\{w\,|\,Im(w)<-\beta_0\}$*, we get*

$$\widetilde{R}^+(w)-\widetilde{R}^-(w)=-2\pi i\frac{e^{-2\pi i\cdot w}}{1-e^{-2\pi i\cdot w}}.$$

Returning to the variable $z=-1/w$ *and to* $H(z)$*, we get*

$$H_+(z)-H_-(z)=2\pi^2 i\frac{e^{-2\pi i\frac{1}{z}}}{1-e^{-2\pi i\frac{1}{z}}}=2\pi^2 if_0(e^{-2\pi i\frac{1}{z}}),\ z\in V^{up},$$

$$H_-(z)-H_+(z)=2\pi^2 i+2\pi^2 i\frac{e^{2\pi i\cdot\frac{1}{z}}}{1-e^{2\pi i\cdot\frac{1}{z}}}=2\pi^2 i+2\pi^2 if_0(e^{2\pi i\cdot\frac{1}{z}}),z\in V^{low}.$$

For the model f_0, the *1-cocycle* of differences $H_+ - H_-$ defines on orbit (quotient) space - see Section 6 for details - the germ $2\pi^2 i f_0$ itself, in both components. This is certainly not a coincidence. It would be interesting to have some geometrical explanation. The differences can be similarly computed by Borel-Laplace transform for any germ f analytically conjugated to f_0. In general, their cocycles are not trivial.

Example 4 (Explicit formulas for the sectorial solutions $H_\pm^{f_0}$ for f_0)**.** *By* (27) *in the proof of Theorem 1, using $f_0^{\circ n}(z) = \frac{z}{1-nz}$, we compute:*

$$H_+^{f_0}(z) = \pi \cdot C^{f_0}(z) - i\pi^2 = \pi \cdot \frac{d}{dz}\log(\Gamma(z))\Big|_{-\frac{1}{z}} - i\pi^2,\ z \in V_+,$$

$$H_-^{f_0}(z) = \pi z - \pi \cdot C^{f_0}(z) = \pi z + \pi \cdot \frac{d}{dz}\log(\Gamma(z))\Big|_{\frac{1}{z}},\ z \in V_-.$$

Here, Γ is the standard Gamma function, holomorphic on $\mathbb{C} \setminus -\mathbb{N}_0$. Therefore, $H_\pm^{f_0}$ are well-defined and analytic on $V_\pm$.

6. Higher-order moments and higher conjugacy classes

We define new classifications of parabolic germs from trivial formal class, based on differences of sectorial solutions of m-Abel equations, as functions on spaces of closed orbits. The well-known Écalle-Voronin moduli are obtained in this manner from Abel equation ($m = 0$), see Subsection 6.1. In particular, in Subsection 6.2, we define a new classification of germs based on 1-Abel equation. We have anticipated in Theorem 2 and supported in Examples 2 and 3 the fact that the trivial class for 1-Abel equation is not related to the trivial analytic class. In Subsection 6.3, we analyse the relative position of analytic classes and classes with respect to 1-Abel equation and get the *transversality* result – they are actually *far away* from each other.

6.1. **Écalle-Voronin moduli of analytic classification.**

This approach to Écalle-Voronin moduli of analytic classification of germs is a reformulation of Fourier representation of moduli from [6, 27] or [5]. The classes are given by pairs of germs at zero, after appropriate identifications. We will use similar approach further in this section to define new classifications imposed by higher Abel equations.

Let $\Psi_+(z),\ z \in V_+$, and $\Psi_-(z),\ z \in V_-$, be sectorial solutions of Abel equation (6) (unique up to additive constant). By pair $\big(h, k\big)$, we denote their differences on V^{up}, V^{low}:

$$h = \Psi_+ - \Psi_-, \ \text{ on } V^{up}, \quad k = \Psi_- - \Psi_+, \ \text{ on } V^{low}.$$

The pair (h,k) is a 1-*cocycle*, in the sense that h and k are analytic germs on petals V^{up}, V^{low}, with exponential decrease, as $z\to 0$.

By equation (6), h and k are constant along closed orbits in V^{up}, V^{low}. We choose positive petal to pass to a space of orbits and represent the space of orbits on V_+ by a punctured sphere, using the change of variables $t=e^{-2\pi i\Psi_+(z)}$ (each orbit corresponds to a point). Closed orbits in V^{up} correspond to a punctured neighborhood of the pole $t=\infty$ and those in V^{low} to a punctured neighborhood of the pole $t=0$. We define a pair of germs $t\mapsto \big(g_\infty(t),g_0(t)\big)$ (on a sphere) around $t=\infty$ and $t=0$ which lift to the pair $\big(h,k\big)$ on the original space:

$$h(z)=g_\infty(e^{-2\pi i\Psi_+(z)}),\ z\in V^{up};\quad k(z)=g_0(e^{-2\pi i\Psi_+(z)}),\ z\in V^{low}.$$

Inverting $g_\infty(t)=g_\infty(1/t)$, g_∞ becomes also a germ at $t=0$. It can be seen that the germs are analytic at punctured neighborhood of 0. They can moreover be extended continuously to 0, by differences of constant terms in sectorial trivialisations. The extension is analytic by Riemann's characterization of removable singularities. We get a *pair of analytic germs* $\big(g_\infty,g_0\big)$ at the origin, satisfying $g_\infty(0)+g_0(0)=0$. Note that the germs are not necessarily diffeomorphisms.

We identify two pairs of germs, (g_∞^1,g_0^1) and (g_∞^2,g_0^2) if it holds that:

$$\begin{aligned} g_\infty^1(0)&=g_\infty^2(0)+a,\quad g_0^1(0)=g_0^2(0)-a,\\ g_\infty^1(t)&=g_\infty^2(bt),\quad g_0^1(t)=g_0^2(t/b), \end{aligned}\tag{34}$$

for $a\in\mathbb{C}$ and $b\in\mathbb{C}^*$. This corresponds to choosing trivialisation functions Ψ_+ and Ψ_- up to an additive constant.

The Écalle-Voronin classification theorem states a *bijective correspondence between analytic classes of germs from the model formal class and all pairs* (g_∞,g_0) *of analytic germs such that* $g_\infty(0)+g_0(0)=0$, *after identifications* (34). The class of germs analytically conjugated to the model is characterized (up to additive constant) by trivial differences $\Psi_+-\Psi_-$ on $V^{up}\cup V^{low}$. It is given by trivial pair of germs $(0,0)$, up to identifications (34). That is, by constant pairs $(-a,a)$, $a\in\mathbb{C}$.

6.2. **Higher moments of classification.** We now mimic the classification technique from Subsection 6.1 on general m-Abel equation:

$$H(f(z))-H(z)=z^m,\ m\in\mathbb{N}_0.\tag{35}$$

By Proposition 1, there exist analytic solutions H_+^m and H_-^m of (35) on petals V_+ and V_-, unique up to a constant term. Their difference on $V^{up}\times V^{low}$ defines a 1-cocycle (h,k), which can further be lifted to the space of orbits of positive sector V_+:

$$h(z) = H_+^m(z) - H_-^m(z) = g_\infty^m(e^{-2\pi i\Psi_+(z)}),\ z \in V^{up},$$
$$k(z) = H_-^m(z) - H_+^m(z) = (-2\pi i) + g_0^m(e^{-2\pi i\Psi_+(z)}),\ z \in V^{low}. \tag{36}$$

The term $-2\pi i$ is put in brackets, since we put it *only in the case when* $m = 1$ for later convenience (to get germs disappearing at 0). In the case $m = 1$, $-2\pi i$ is the difference of the two branches of logarithm.

As above, we get a pair of germs $t \mapsto (g_\infty^m(t), g_0^m(t))$ on a neighborhood of origin, that can be extended analytically to $t = 0$.

The trivialisation Ψ_+ is determined only up to an arbitrary constant. Also, if we add any complex constant to H_+^m or H_-^m, they remain solutions of m-Abel equation (35). Thus, as before, we identify two pairs of germs if (34) holds. Again, $g_0^m(0) + g_\infty^m(0) = 0$.

Definition 4 (The m-moments)**.** *Let* $m \in \mathbb{N}_0$. *The* m-moment of a germ f with respect to trivialization function of the attracting petal *or, shortly,* m-moment of f, *is the pair*

$$t \mapsto \big(g_\infty^m(t),\ g_0^m(t)\big)$$

of analytic germs at zero from (36)*, up to identifications* (34)*.*

The definition of *g-moment* can be formulated for any right-hand side $g \in \mathbb{C}\{z\}$ of the equation (7). The similar definition is used in [7, Section 4] to express the second pair of moduli of analytic classification of 2-dimensional t-shifts. For more information, see Subsection 7.2.

Remark 1. *The* 1*-moments are defined using differences of sectorial solutions* $R_+ - R_-$ *on* $V^{up} \cap V^{low}$ *of the modified equation*

$$R(f(z)) - R(z) = z - \log\big(f(z)/z\big),$$

instead of differences $H_+ - H_-$ *from the original equation* (37)*. The equation is obtained substituting* $H(z) = \log z + R(z)$ *in* (37)*. Thus we remove the constant term* $-2\pi i$ *from* (36)*.*

Definition 5 (The m-conjugacy relation for parabolic germs)**.** *Let* $m \in \mathbb{N}_0$. *The* m-conjugacy *is the equivalence relation on the set of all germs from the model formal class, given by*

$$f_1 \overset{m}{\sim} f_2, \text{ if and only if } f \text{ and } g \text{ have the same } m\text{-moments.}$$

By $[f]_m = \{g \mid g \overset{m}{\sim} f\}$*, we denote the* m*-conjugacy class of* f*.*

Instead of equation (5), we consider here the 1-Abel equation:

$$H(f(z)) - H(z) = z. \tag{37}$$

The moments are scaled by -$\frac{1}{\pi}$. The conjugacy classes remain the same.

Example 5 (0- and 1-conjugacy classes)**.**

(1) *The Abel equation being* 0*-Abel equation, the* 0*-moments correspond to Écalle-Voronin moduli from Subsection 6.1 and the* 0*-conjugacy classes to the analytic classes. In particular, the germs analytically conjugated to the model have the pair* $(0,0)$ *as* 0*-moment* (*the Abel equation has global solution*)*.*

(2) *The* 1*-conjugacy classes are obtained from* 1*-Abel equation. By Theorem 2, the trivial* 1*-conjugacy class* (*the set of all germs with* 1*-moments equal to* $(0,0)$*, i.e., with global solution to equation* (37) *or, equivalently, to* (5)) *is the set*

$$\mathcal{S}=\Big\{f\mid f=\varphi^{-1}(e^z\cdot\varphi(z)),\ \varphi\in z+z^2\mathbb{C}\{z\}\Big\}.$$

We complete the section with question of *realization* of moments. All possible 0- or 1-conjugacy classes are given by all pairs of analytic germs (g_1,g_2) at zero, after identifications (34), such that $g_1(0)+g_2(0)=0$.

Proposition 7 (Realization of 0-moments)**.** *For every pair* (g_1,g_2) *of analytic germs, such that* $g_1(0)+g_2(0)=0$*, there exists a germ* f *from model formal class, such that the pair* $\big(g_1,g_2\big)$ *is its* 0*-moment.*

Proof. The statement follows directly from the theorem of realization of Écalle-Voronin moduli, see [6, 27] or [5, Theorem 18]. □

Proposition 8 (Realization of 1-moments)**.** *For every pair* (g_1,g_2) *of analytic germs, such that* $g_1(0)+g_2(0)=0$*, there exists a germ* f *from model formal class, such that the pair* $\big(g_1,g_2\big)$ *is its* 1*-moment.*

Note that by varying constant term chosen in sectorial trivialisation function Ψ_+ and constants chosen in solutions H_+ and H_-, we can realise all other 1-moments identified by (34) by the same germ f.

Proof. See the proof of Theorem 4 in Subsection 6.3 below. □

6.3. **Relative position of** 1**-conjugacy and analytic classes.**

Let Φ denote the mapping

$$\Phi(f)=[f]_1,$$

defined on the set of all germs from model formal class. We precisely state and prove Theorem 4 from Section 1:

Theorem 4 (*Transversality*)**.** *The restriction* $\Phi\Big|_{[f]_0}$ *maps surjectively from any analytic class* $[f]_0$ *onto the set of all* 1*-conjugacy classes.*

We first give an outline of the proof. We prove not only that every pair of germs can be realized as 1-moment in the model formal class (Proposition 8), but also as 1-moment in *any* analytic class (Theorem 4). Take any analytic class $[f]_0$ and a representative f. Let (g_1, g_2) be any pair of analytic germs, $g_1(0) + g_2(0) = 0$. We first show that there exists an analytic, tangent to the identity right-hand side δ of the cohomological equation for f, such that (g_1, g_2) is the moment of f with respect to this equation. This idea is *borrowed* from [14, A.6]. Then, by change of variables, we transform the equation to 1-Abel equation for a different germ, analytically conjugated to f by δ.

Proof of Proposition 8 and of Theorem 4. Let $[f]_0$ be any analytic class and $f \in [f]_0$. Let $\Psi_+^f(z)$ be any trivialisation of V_+ for f. Let (g_1, g_2) be any pair of analytic germs, satisfying $g_1(0) + g_2(0) = 0$.

On some petals V^{up} and V^{low} of opening π and centered at directions $\pm i$ respectively, we define the pair $\big(T_\infty, T_0\big)$ by:

(38)
$$T_\infty(z) = g_1(e^{2\pi i \Psi_+^f(z)}),\ z \in V^{up};\quad T_0(z) = g_2(e^{-2\pi i \Psi_+^f(z)}),\ z \in V^{low}.$$

If $g_1(0),\ g_2(0) \neq 0$, we subtract the constant term. This can be done without loss of generality, since a constant term can be added to sectorial solutions afterwards. By construction, T_∞ and T_0 are f-invariant.

The functions $z \mapsto T_0(z)$ and $z \mapsto T_\infty(z)$ are analytic and exponentially decreasing of order one on V^{up} and V^{low}. The pair (T_∞, T_0) defines a 1-cocycle. By Ramis-Sibuya theorem, see e.g. [14, Théorème] or [3, Theorem 2.5], [25], there exists a formal series $\widehat{H} \in z\mathbb{C}[[z]]$, which is 1-summable[2] along arcs of directions $\theta \in (\pi/2, 3\pi/2)$ and $\theta \in (-\pi/2, \pi/2)$, and whose differences of 1-sums H_+ and H_- thus defined on petals[3] V_+ and V_- realize the cocycle (T_∞, T_0):

$$T_0 = H_+ - H_- \text{ on } V^{up},\ T_\infty = H_- - H_+ \text{ on } V^{low}. \tag{39}$$

We adapt now slightly functions H_+ and H_- by adding the appropriate branch of logarithm,

$$\widetilde{H}_\pm(z) = H_\pm(z) + \log(z),\ z \in V_\pm. \tag{40}$$

[2]To recall, we say that $\widehat{H}(z)$ is 1-*summable* in the direction $\theta = \theta_0$ if there exists an analytic function $H(z)$, defined on some sector V of opening at least π and centered at $\theta = \theta_0$, such that H admits $\widehat{H}$ as its asymptotic expansion on V, as $z \to 0$, which additionally satisfies some uniform Gevrey bounds, see e.g. [20, Section 2.1] or [3] for precise definitions. H is then called a *1-sum* of $\widehat{H}$ on V.

[3]of opening 2π and centered at $\theta = \pi$ and $\theta = 0$ respectively. Petals are unions of sectors with decreasing radii while increasing opening.

We define functions $\delta_\pm$ on $V_\pm$ respectively by:

$$\begin{aligned} &\delta_+(z) = \widetilde{H}_+(f(z)) - \widetilde{H}_+(z),\ z \in V_+,\\ &\delta_-(z) = \widetilde{H}_-(f(z)) - \widetilde{H}_-(z),\ z \in V_-. \end{aligned} \tag{41}$$

From (39) and (41), using f-invariance of T_∞ and T_0 and Riemann's theorem on removable singularities, we see that δ_+ and δ_- glue to an analytic germ δ. By (40), $\delta \in z + z^2\mathbb{C}\{z\}$. To conclude, $\widetilde{H}_+$ and $\widetilde{H}_-$ are sectorial solutions of the cohomological equation for germ f, with the right-hand side δ. That is,

$$\widetilde{H} \circ f - \widetilde{H} = \delta.$$

By analytic change of variables $w = \delta(z)$, we get

$$\widetilde{H} \circ \delta^{-1}(\delta \circ f \circ \delta^{-1}(w)) - (\widetilde{H} \circ \delta^{-1})(w) = w.$$

Therefore, $(\widetilde{H} \circ \delta^{-1})_\pm(z) = (\widetilde{H}_\pm \circ \delta^{-1})(z)$, $z \in V_\pm$ ($V_\pm$ being in fact $\delta(V_\pm)$, but identified with $V_\pm$ since δ is a conformal map tangent to the identity) are solutions of 1-Abel equation for germ $g = \delta \circ f \circ \delta^{-1}$, analytically conjugated to f. Furthermore, by (38) and (39),

$$\begin{aligned} &(\widetilde{H}_+ \circ \delta^{-1})(z) - (\widetilde{H}_- \circ \delta^{-1})(z) = T_\infty(\delta^{-1}(z)) =\\ &\qquad = g_1(e^{2\pi i \Psi_+^f \circ \delta^{-1}(z)}) = g_1(e^{2\pi i \Psi_+^g}(z)),\ z \in V^{up},\\ &(\widetilde{H}_- \circ \delta^{-1})(z) - (\widetilde{H}_+ \circ \delta^{-1})(z) = -2\pi i + T_0(\delta^{-1}(z)) =\\ &\qquad = -2\pi i + g_2(e^{-2\pi i \Psi_+^f \circ \delta^{-1}(z)}) = -2\pi i + g_2(e^{-2\pi i \Psi_+^g}(z)),\ z \in V^{low}. \end{aligned}$$

Here, $\Psi_+^g(z) = \Psi_+^f \circ \delta^{-1}$ is a trivialisation function for g, for an appropriate choice of constant term. Thus, the cocycle (g_1, g_2) is realized as 1-moment of germ g, analytically conjugated to f. □

We pose the question of *injectivity* in Theorem 4. Inside any analytic class we can construct different germs with the same 1-moment:

Proposition 9 (Non-injectivity)**.** *Let $f,\ g \in [f]_0$. If there exists a change of variables $\varphi \in z + z^2\mathbb{C}\{z\}$, $g = \varphi^{-1} \circ f \circ \varphi$, of the form*

$$\varphi^{-1} = Id + r \circ f - r, \tag{42}$$

for some $r \in \mathbb{C}\{z\}$, then f and g have the same 1-moments. Additionally, if $[f]_0$ is the model analytic class, the implication takes the form of the equivalence statement.

The proof can be found in preprint [22].

For future research, we pose the question of relative position of higher equivalence classes to each other: if *transversality* holds as a rule.

7. Perspectives

7.1. **Can we recognize a parabolic diffeomorphism using directed areas of ε-neighborhoods of *only one* orbit?**

We use ideas from proof of Proposition 4 to prove that a germ f is uniquely determined by the function $\varepsilon \mapsto A^{\mathbb{C}}(z_0,\varepsilon)$, $\varepsilon\in(0,\varepsilon_0)$, where $\varepsilon_0>0$ is arbitrary small and z_0 fixed belongs to the basin of attraction of f. Since this function is defined using *only one orbit*, the areas of ε-neighborhoods of only one orbit should be enough to read the analytic class. How this can be done, remains subject to further research. This is a different approach to the problem; in the article, we have been considering sectorial functions, related to $A^{\mathbb{C}}(z,\varepsilon)$, in variable z.

Let $\mathrm{Diff}\,(\mathbb{C},0;z_0)\subset \mathrm{Diff}\,(\mathbb{C},0)$ denote the set of all parabolic germs whose basin of attraction contains z_0.

Proposition 10. *Let $\varepsilon_0>0$, z_0 be fixed. The mapping*

$$f\in \mathit{Diff}\,(\mathbb{C},0;z_0)\longmapsto \big(\varepsilon\mapsto A^{\mathbb{C}}(z_0,\varepsilon),\ \varepsilon\in(0,\varepsilon_0)\big)$$

is injective on $\mathit{Diff}\,(\mathbb{C},0;z_0)$.

Proof. Suppose that $A^{\mathbb{C},f}(z_0,\varepsilon)=A^{\mathbb{C},g}(z_0,\varepsilon)$, $\varepsilon\in(0,\varepsilon_0)$, for some $f,\ g\in\mathrm{Diff}\,(\mathbb{C},0;z_0)$. We show that the germs f and g must be equal.

Separating the tails and the nuclei and dividing by $\varepsilon^2\pi$, we get

$$\frac{A^{\mathbb{C},f}(T_\varepsilon)-A^{\mathbb{C},g}(T_\varepsilon)}{\varepsilon^2\pi}=\frac{A^{\mathbb{C},g}(N_\varepsilon)-A^{\mathbb{C},f}(N_\varepsilon)}{\varepsilon^2\pi},\ \varepsilon\in(0,\varepsilon_0). \tag{43}$$

The proof relies on presence of singularities of directed areas at $(\varepsilon_n^{f,g})$. Let $z_n,\ w_n,\ n\in\mathbb{N}$, denote the points of the orbits $S^f(z_0)$ and $S^g(z_0)$ respectively. Recall that

$$\varepsilon_n^f=\frac{|z_n-z_{n+1}|}{2},\ \varepsilon_n^g=\frac{|w_n-w_{n+1}|}{2},\quad n\in\mathbb{N}.$$

Suppose that the sequences of singularities for f and g, (ε_n^f) and (ε_n^g), do not eventually coincide. Then there exist $m,\ n\in\mathbb{N}$ arbitrary big and an interval $(\varepsilon_n^f-\delta,\varepsilon_n^f+\delta)$, $\delta>0$, such that $\varepsilon_m^g<\varepsilon_n^f-\delta$ and $\varepsilon_n^f+\delta<\varepsilon_{m-1}^g$. Consider the second derivative $\frac{d^2}{d\varepsilon^2}$ of (43) at ε_n^f from the right. With the notations as in proof of Proposition 4, by (50),

$$\begin{aligned}0=&(G_{n+1}^f)''(\varepsilon_n^f+)-(G_m^g)''(\varepsilon_n^f+)+\\ &+\frac{1}{\pi}\left(\frac{4\varepsilon_n^f}{\varepsilon^3}\sqrt{1-\frac{(\varepsilon_n^f)^2}{\varepsilon^2}}-\frac{2(\varepsilon_n^f)^3}{\varepsilon^5}\frac{1}{\sqrt{1-\frac{(\varepsilon_n^f)^2}{\varepsilon^2}}}\right)\Bigg|_{\varepsilon=\varepsilon_n^f+}\cdot\big(z_{n+1}+z_n\big).\end{aligned}\tag{44}$$

Since all terms are bounded except the term in brackets and $z_n+z_{n+1}\neq 0$, (44) leads to a contradiction. Therefore, sequences of singularities (ε_n^f) and (ε_n^g) eventually coincide,

$$\varepsilon_n^f=\varepsilon_{n+k_0}^g,\ n\geq n_0,\ k_0\in\mathbb{N}.$$

Now, considering the second derivative of (43) at $\varepsilon_n=\varepsilon_n^f=\varepsilon_{n+k_0}^g$ from the right, we get:

$$0=\big(G_{n+1}^f\big)''(\varepsilon_n+)-\big(G_{n+k_0+1}^g\big)''(\varepsilon_n+)+$$
$$+\frac{1}{\pi}\left(\frac{4\varepsilon_n}{\varepsilon^3}\sqrt{1-\frac{\varepsilon_n^2}{\varepsilon^2}}-\frac{2\varepsilon_n^3}{\varepsilon^5}\frac{1}{\sqrt{1-\frac{\varepsilon_n^2}{\varepsilon^2}}}\right)\Bigg|_{\varepsilon=\varepsilon_n+}\cdot\Big(z_{n+1}+z_n-\big(w_{n+k_0+1}+w_{n+k_0}\big)\Big).$$

The term in brackets is the only unbounded term, so $(z_{n+1}+z_n)-(w_{n+k_0+1}+w_{n+k_0})=0$. The middle points of orbits $S^f(z_0)$ and $S^g(z_0)$ eventually coincide. The distances $d_n^f=2\varepsilon_n^f$ and $d_n^g=2\varepsilon_n^g$ coincide, and both orbits converge to some tangential direction. It is easy to see that the orbits themselves eventually coincide. Two analytic germs f and g coincide on a set accumulating at the origin, so they are equal. $\square$

7.2. **Cohomological equations in analytic classification of 2-dimensional t-shifts.**

This result is due to David Sauzin in personal communication. It gives an application of cohomological equations for one-dimensional germs f, along with their Abel equation, to analytic classification of two-dimensional t-shifts, $F:\big((\mathbb{C},0),\mathbb{C}\big)\to\big((\mathbb{C},0),\mathbb{C}\big)$,

$$F(z,t)=\big(f(z),t+a(z)\big),\ a(z)\in\mathbb{C}\{z\}.$$

The analytic classification of t-shifts is discussed in [7], in view of analytic classification of resonant complex saddles. They appear as saddle t-monodromies, the parabolic germ f being a holonomy map of the saddle and t the complex return time. In [7, Section 4.2], the four moduli of analytic classification are deduced without mention of cohomological equations. On the other hand, they agree with the quadruple containing the 0-moment and the a-moment for f, defined in Section 8.

Let f belong to the formal class of f_0. The t-shift above with $a(z)=\alpha_0+\alpha_1 z+o(z)$ can by (t-shift) formal change of variables be reduced to a formal normal form

$$F_0(z,t)=(f_0(z),\alpha_0+\alpha_1 z+t).$$

Equivalently, we search for formal solutions $\widehat{T}(z,t)$ of the trivialization equation:

$$\widehat{T}(F(z,t))=\widehat{T}(z,t)+(1,0). \tag{45}$$

It can be computed that the formal trivialization $\widehat{T}$ for F is given by

$$\widehat{T}(z,t)=\Big(\widehat{\Psi}(z),\widehat{H}^{-a}(z)+t\Big). \tag{46}$$

Here, $\widehat{\Psi}$ is the formal solution of the Abel equation for f, and $\widehat{H}^{-a}$ of the cohomological $(-a)$-equation for f. Hence, Abel equation appears as the first and $(-a)$-equation as the second coordinate in the trivialization equation (45) for t-shift F. Accordingly, it can be compared with [7] that the analytic moduli of F from [7] are in fact given by moments (as defined in Subsection 6.2) with respect to both equations.

8. Appendix

Lemma 1 (Asymptotic expansions for germs of multiplicity $k=1$)**.** *Let $f(z)=z+a_1z^2+a_2z^3+o(z^3)$ be a parabolic germ and z_0 an initial point. Then we have the following expansions:*

(1) $f^{\circ n}(z_0)=-\frac{1}{a_1}n^{-1}+\Big(\frac{1}{a_1}-\frac{a_2}{a_1^3}\Big)n^{-2}\log n+q(z)\cdot n^{-2}+o(n^{-2}),\ n\to\infty.$

(2) $n_\varepsilon=(2|a_1|)^{-1/2}\varepsilon^{-1/2}+\frac{1}{2}Re\big(1-\frac{a_2}{a_1^2}\big)\log\varepsilon+o(\log\varepsilon),\ \varepsilon\to 0.$

(3) (*Center of the mass of the nucleus*)

$$A(N_\varepsilon)t(N_\varepsilon)=-\frac{1}{a_1}\cdot\frac{\pi}{4}(1+\log 4)\cdot\varepsilon^2-$$
$$-\frac{\sqrt{\pi}}{8a_1}(2|a_1|)^{\frac12}\left(\sqrt{\pi}-\frac{\Gamma(3/4)}{\Gamma(5/4)}\right)\cdot i\cdot Im\big(1-\frac{a_2}{a_1^2}\big)\varepsilon^{\frac52}\cdot\log\varepsilon+o(\varepsilon^{\frac52}\log\varepsilon),\ \varepsilon\to 0.$$

(4) (*Center of the mass of the tail*)

$$A(T_\varepsilon)t(T_\varepsilon)=\frac{\pi}{2a_1}\varepsilon^2\log\varepsilon+H_T^f(z_0)\varepsilon^2-$$
$$-\pi\left(\frac{|a_1|}{2}\right)^{1/2}\frac{1}{a_1}\cdot i\cdot Im\big(1-\frac{a_2}{a_1^2}\big)\varepsilon^{\frac52}\log\varepsilon+o(\varepsilon^{\frac52}\log\varepsilon),\ \varepsilon\to 0.$$

Proof. In [21, Lemmas 1-5, Proposition 3], expansions are given for germs of multiplicity $k>1$. First, (1) is obtained as in [21, Proposition 3], putting $k=1$, but iterating further to get more terms of the expansion. To get expansions (2), (3), (4), we insert $k=1$ in the coefficients from [21, Lemmas 1-3]. On the contrary, in [21, Lemmas 4 and 5], the first coefficients diverge for $k=1$. In the case $k=1$, the first formula for q_1 is different: in the proof of [21, Lemma 4], it is computed evaluating the integral $\int_0^1(t\sqrt{1-t^2}+\arcsin t)t^{-1}\,dt$, whose formula differs from integrals $\int_0^1(t\sqrt{1-t^2}+\arcsin t)\cdot t^{-a}\,dt,\ 0<a<2,\ a\neq 1.$

Further, in the proof of [21, Lemma 5] in the case $k=1$, the expansion begins with the logarithmic term that was previously the k-th term. $\square$

Proof of Proposition 3. We show the obstacle for the existence of a full expansion: the index n_ε separating the tail and the nucleus of the ε-neighborhood does not have expansion in ε after the first $k+1$ terms.

We need here a refinement of the expansion from [21, Lemma 1]:

$$\begin{aligned} n_\varepsilon = p_1\varepsilon^{-1+\frac{1}{k+1}} + \ldots + p_k\varepsilon^{-1+\frac{k}{k+1}} + p_{k+1}\log\varepsilon + r(z,\varepsilon), \\ r(z,\varepsilon) = o(\log\varepsilon),\ \varepsilon \to 0. \end{aligned} \tag{47}$$

We compute one more term of the expansion of z_n from [21, Prop. 3]:

$$\begin{aligned} z_n =& g_1 n^{-\frac{1}{k}} + g_2 n^{-\frac{2}{k}} + g_3 n^{-\frac{3}{k}} + g_4 n^{-\frac{4}{k}} + \ldots + \\ &+ g_k n^{-1} + g_{k+1} n^{-\frac{k+1}{k}} \log n + q(z)\cdot n^{-\frac{k+1}{k}} + o(n^{-\frac{k+1}{k}}),\ n \to \infty. \end{aligned}$$

By the same procedure from the proof of [21, Lemma 1], due to one more term in z_n and thus also in $d_n = |z_{n+1} - z_n|$, in (47) we get the refinement $r(z,\varepsilon) = O(1)$ in ε. We write z only to denote dependence of $r(z,\varepsilon)$ on the initial point. Here, z is only a fixed complex number.

Suppose that $\lim_{\varepsilon\to 0} r(z,\varepsilon)$ exists. Then,

$$r(z,\varepsilon) = C(z) + o(1),\ \varepsilon \to 0 \quad (C \text{ can be } 0). \tag{48}$$

In the points ε_n as above, it holds that $n(\varepsilon_n+) = n,\ n(\varepsilon_n-) = n+1$. The $(k+1)$-jet in expansion (47) is continuous on $(0,\varepsilon_0)$. By (48), $r(\varepsilon_n) = C + o(1)$, as $n \to \infty$. Therefore we get that

$$1 = n(\varepsilon_n+) - n(\varepsilon_n-) = o(1),\ n \to \infty,$$

which is a contradiction. The limit $\lim_{\varepsilon\to 0} r(z,\varepsilon)$ does not exist.

Now, in the proofs of Lemmas 4, 5 in [21] for the tail and the nucleus, we check that $A^{\mathbb{C}}(z,\varepsilon)$ in general does not have full expansion in ε. $\square$

Proof of Proposition 4. We consider the directed area divided by $\varepsilon^2\pi$. We show that the points where class C^2 is lost are points ε_n where, as ε decreases, one disc detaches from the nucleus to the tail. We have

$$\frac{A^{\mathbb{C}}(z,\varepsilon)}{\varepsilon^2\pi} = \frac{A^{\mathbb{C}}(T_\varepsilon)}{\varepsilon^2\pi} + \frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}.$$

The tail function $\varepsilon \mapsto \frac{A^{\mathbb{C}}(T_\varepsilon)}{\varepsilon^2\pi}$ is a piecewise constant function on $[\varepsilon_{n+1},\varepsilon_n)$, $n\in\mathbb{N}$, with jumps at ε_n of value z_n. The area of the nucleus is the sum of contributions of crescents. By Proposition 5 in [21],

(49)
$$\frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}=\begin{cases} z_{n+1}+G_{n+1}(\varepsilon), & \varepsilon\in[\varepsilon_{n+1},\varepsilon_n),\\ z_n+G_{n+1}(\varepsilon)+ & \\ \quad +\frac{1}{\pi}\left(\frac{\varepsilon_n}{\varepsilon}\sqrt{1-\frac{\varepsilon_n^2}{\varepsilon^2}}+\arcsin\frac{\varepsilon_n}{\varepsilon}\right)(z_n+z_{n+1})+\frac{z_{n+1}-z_n}{2}, & \\ & \varepsilon\in[\varepsilon_n,\varepsilon_{n-1}).\end{cases}$$

Here, $G_{n+1}(\varepsilon)$ is computed as the sum of contributions of the crescents corresponding to the points $z_{n+2},\ z_{n+3}$, etc.:

$$G_{n+1}(\varepsilon)=\frac{1}{\pi}\sum_{k=n+1}^{\infty}\left(\frac{\varepsilon_k}{\varepsilon}\sqrt{1-\frac{\varepsilon_k^2}{\varepsilon^2}}+\arcsin\frac{\varepsilon_k}{\varepsilon}\right)(z_k+z_{k+1})+\frac{z_{k+1}-z_k}{2}.$$

Let $\delta>0$ such that $\varepsilon_{n+1}+\delta<\varepsilon_n$. It can be checked that G_{n+1} is a well-defined C^∞-function on $(\varepsilon_{n+1}+\delta,\varepsilon_{n-1})$, and the differentiation is performed term by term. By (49), the singularity of $A^{\mathbb{C}}(N_\varepsilon)$ on $(\varepsilon_{n+1}+\delta,\ \varepsilon_{n-1})$ can only be point $\varepsilon=\varepsilon_n$, where two parts defined by different formulas are glued together. To check, we differentiate (49) twice in ε on some interval around ε_n:

$$\frac{d}{d\varepsilon}\frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}\Big|_{\varepsilon=\varepsilon_n-}=G'_{n+1}(\varepsilon_n-),\ \ \frac{d}{d\varepsilon}\frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}\Big|_{\varepsilon=\varepsilon_n+}=G'_{n+1}(\varepsilon_n+),$$

Both derivatives are finite and equal since G_{n+1} is of the class C^2 around ε_n. Therefore, $A^{\mathbb{C}}(N_\varepsilon)$ is of class C^1 at $\varepsilon=\varepsilon_n$, $n\in\mathbb{N}$. However,

$$\frac{d^2}{d\varepsilon^2}\frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}\Big|_{\varepsilon=\varepsilon_n-}=(G_{n+1})''(\varepsilon_n-),$$
$$\frac{d^2}{d\varepsilon^2}\frac{A^{\mathbb{C}}(N_\varepsilon)}{\varepsilon^2\pi}\Big|_{\varepsilon=\varepsilon_n+}=(G_{n+1})''(\varepsilon_n+)+$$
$$+\frac{1}{\pi}\Big(\frac{4\varepsilon_n}{\varepsilon^3}\sqrt{1-\frac{\varepsilon_n^2}{\varepsilon^2}}-\frac{2\varepsilon_n^3}{\varepsilon^5}\frac{1}{\sqrt{1-\frac{\varepsilon_n^2}{\varepsilon^2}}}\Big)\Big|_{\varepsilon=\varepsilon_n+}\cdot\big(z_{n+1}+z_n\big). \tag{50}$$

Although $(G_{n+1})''(\varepsilon_n-)=(G_{n+1})''(\varepsilon_n+)\in\mathbb{C}$, the other term is unbounded. The second derivative of $A^{\mathbb{C}}(N_\varepsilon)$ at $\varepsilon=\varepsilon_n$, $n\in\mathbb{N}$, does not exist and the class C^2 is lost at ε_n. Finally, glueing overlapping intervals $(\varepsilon_{n-1}+\delta,\varepsilon_{n+1}),\ n\in\mathbb{N}$, we get the desired result. □

Proof of Proposition 5. For $\varepsilon>0$, let $U_\varepsilon=\{z\in V_+:\ |z-f(z)|<2\varepsilon\}$. For $z\in U_\varepsilon$, the ε-discs centered at points z and $f(z)$ in $S^f(z)_\varepsilon$ overlap.

Therefore, by Proposition 5 in [21],

$$A^{\mathbb{C}}(z,\varepsilon)=A^{\mathbb{C}}(f(z),\varepsilon)-\frac{\pi}{2}\varepsilon^2(f(z)-z)+ \\ +\varepsilon^2(z+f(z))\cdot G\Big(\frac{|z-f(z)|}{2\varepsilon}\Big),\ z\in U_\varepsilon. \tag{51}$$

Here, $G(t)=t\sqrt{1-t^2}+\arcsin t,\ t\in(0,1)$. We define function T:

$$T(z)=A^{\mathbb{C}}(z,\varepsilon)-A^{\mathbb{C}}(f(z),\varepsilon),\ z\in V_+. \tag{52}$$

By (51), it holds

$$T(z)=-\frac{\pi}{2}\varepsilon^2(f(z)-z)+\varepsilon^2(z+f(z))\cdot G\Big(\frac{|z-f(z)|}{2\varepsilon}\Big),\ z\in U_\varepsilon.$$

There exists a punctured neighborhood of 0 such that $f'(z)\neq 1$, for all z. Otherwise, by analyticity of f, $f'\equiv 1$ on some neighborhood of 0, not true. By inverse function theorem applied locally to G and $\mathrm{Id}-f$, since absolute value is nowhere analytic, T is nowhere analytic on U_ε.

Take any sector $S^+(\varphi,r)\subset V_+,\ r>0,\ \varphi\in(0,\pi)$. Suppose that $z\mapsto A^{\mathbb{C}}(z,\varepsilon)$ is analytic on S_+. Since f is analytic, and $f(z)\in S_+$ for $z\in S_+$, the function $z\mapsto T(z)$ from (52) is analytic on S_+. The intersection $S_+\cap U_\varepsilon$ is nonempty and we derive a contradiction. $\square$

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References

[1] P. Ahern, J.-P. Rosay, *Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables*, Trans. Amer. Math. Soc. 347 (1995), no. 2, p. 543–572

[2] G. Belitskii, V. Tkachenko, *One-dimensional functional equations*, Operator Theory, Advances and Applications, Vol. 144, Birkhäuser Verlag (2003)

[3] M. Canalis-Durand, *Asymptotique Gevrey*, RTNS lecture notes, Segovia, 25 January 2010 - 29 January 2010

[4] B. Candelpergher, *Une introduction à la résurgence*, Gaz.Math.42,36-64 (1989)

[5] A. Dudko, *Dynamics of holomorphic maps: Resurgence of Fatou coordinates, and poly-time computability of julia sets*, thesis, University of Toronto (2012).

[6] J. Écalle, *Les functions résurgentes*, Publ. Math. d'Orsay, Vol. 1: 81-05, Vol. 2: 81-06, Vol. 3:85-05, 1981, 1985.

[7] A. A. Grintchy, S. M. Voronin, *An analytic classification of saddle resonant singular points of holomorphic vector fields in the complex plane*, J. Dynam. Control Systems 2 (1996), no. 1, 21–53

[8] L. Horvat Dmitrović, *Box dimension and bifurcations of one-dimensional discrete dynamical systems*, Discrete Contin. Dyn. Syst. **32**, 4 (2012), 1287–1307

[9] Y. Ilyashenko, S. Yakovenko, *Lectures on Analytic Differential Equations, Graduate Studies in Mathematics*, 86. American Mathematical Society, Providence, RI, xiv+625 pp, 2008.

[10] M. L. Lapidus, E. P. J. Pearse, *Tube formulas and complex dimensions of self-similar tilings*, Acta Appl. Math. **112**, 1 (2010), 91–136

[11] M. L. Lapidus, E. P. J. Pearse, *A tube formula for the Koch snowflake curve, with applications to complex dimensions*, J. Lond. Math. Soc., II. Ser. 74, No. 2 (2006), 397–414

[12] M. L. Lapidus, C. Pomerance, *The Riemann Zeta-function and the one-dimensional Weyl-Berry Conjecture for fractal drums*, Proc. London Math.Soc. 66, 1 (1993), 41–69

[13] A. N. Livšic, *Cohomology of dynamical systems*, Math. U.S.S.R. Izvestia 6 (1972), 1278–1301

[14] F. Loray, *Analyse des séries divergentes*, dans Quelques aspects des mathématiques actuelles, Mathématiques 2eme cycle, Ellipses (1998), 111–173

[15] F. Loray, *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*, Prépublication IRMAR (2005). ccsd-00016434.

[16] Yu. I. Lyubich, *Axiomatic theory of divergent series and cohomological equations*, Fund. Math. 198 (2008), no. 3, 263-282

[17] P. Mardešić, M. Resman, V. Županović, *Multiplicity of fixed points and ε-neighborhoods of orbits*, J. Differ. Equ. 253 (2012), 2493–2514

[18] J. Milnor, *Dynamics in one complex variable, Introductory Lectures, 2nd Edition* (2000), Friedr. Vieweg. & Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden.

[19] J. Palis, F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press (1993)

[20] J. P. Ramis, *Séries divergentes et théories asymptotiques*, Paris : Société mathématique de France, Series: Panoramas et synthèses (1993)

[21] M. Resman, *ε-neighborhoods of orbits and formal classification of parabolic diffeomorphisms*, Discr. Cont. Dyn. Syst., Series A, 33(8) (2013), 3767–3790

[22] M. Resman, *Epsilon-neighborhoods of orbits of parabolic diffeomorphisms and cohomological equations*, a preprint (2013), arXiv:1307.0780v3

[23] M. Resman, *Fixed points of diffeomorphisms, singularities of vector fields and epsilon-neighborhoods of their orbits, the thesis* (2013), arXiv:1311.2170v1.

[24] D. Sauzin, *Resurgent functions and splitting problems*, Preprint, CNRS-IMCCE, Paris (2006).

[25] Y. Sibuya, *Linear differential equations in the complex domain, problems of analytic continuation*, Trans. Math. Monographs vol 82 (Providence, RI: Ann. Math. Soc.) (1990)

[26] C. Tricot, *Curves and Fractal Dimension*, Springer-Verlag, New York (1993)

[27] S. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbb{C},0) \to (\mathbb{C},0)$, Functional Anal. Appl. 15, no. 1, 1.13.(1981)

[28] V. Županović, D. Žubrinić, *Fractal dimension in dynamics*, Encyclopedia of Math. Physics, vol. 2, Elsevier, Oxford (2006)

[29] D. Žubrinić, V. Županović, *Poincaré map in fractal analysis of spiral trajectories of planar vector fields*, Bull. Belg. Math. Soc. S. Stevin, 15 (2008) 947–960

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