DERIVATION OF HOMOGENIZED EULER-LAGRANGE EQUATIONS FOR INEXTENSIBLE VON KÁRMÁN ROD

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Abstract. In this paper we study the effects of simultaneous homogenization and dimension reduction in the context of convergence of stationary points for thin inextensible nonhomogeneous rods under the assumption of the von Kármán scaling regime. Assuming stationarity condition for a sequence of deformations close to a rigid body motion, we prove that the corresponding sequences of scaled displacements and twist functions converge to a limit point, which is the stationary point of the homogenized von Karman rod model. The analogous result holds true for the von Karman plate model.

1. Introduction

Boosted by the rigidity result of Friesecke, James and Múller [12], rigorous derivation of various approximate models from three-dimensional nonlinear elasticity theory and its variational justification has become a prominent research topic in the last decade. In particular, based on a refined rigidity result [13], a whole hierarchy of limiting lower-dimensional models has been derived by means of Γ-convergence techniques [4, 9]. For the context reasons, we only refer to the derivation of nonlinear inextensible rod models [20, 22]. However, in all these models material is assumed to be homogeneous. There is a vast literature on studying the effects of simultaneous homogenization and dimension reduction in various contexts [5, 8, 17], but again we focus on rod models. First attempts date back to [18], where the authors studied a linearized rod model assuming its homogeneity along the central line and nonhomogeneous microstructure in the cross section. A systematic approach combining rigidity estimates [13] and the two-scale convergence method [1] (periodicity assumption) was presented in [24] for the model of homogenized bending rod. The same model has been obtained in [19] without periodicity assumptions, while using a Γ-convergence method tailored to the dimension reduction in higher-order elasticity models, which also applies for the derivation of homogenized von Kármán plate [27] and linearized elasticity models [7]. In this
The main purpose of this paper is to study convergence of stationary points of thin three-dimensional inhomogeneous rods in the von Kármán scaling regime. The above mentioned Γ-convergence techniques roughly assert that a compact sequence of minimizers of scaled energies converges (on a subsequence) to a minimizer of the limit energy. However, due to nonlinearities, these minimizers are typically only global and do not necessary satisfy the corresponding Euler–Lagrange equation. Convergence of stationary points of thin elastic rods in the bending regime has been first studied in [23] on a simplified model of thin 2D strips and thenafter extended to the full 3D problem in [21]. In order to identify the limit equations, besides the rigidity estimate, the authors also used compensated compactness and careful truncation arguments. Later on, convergence of stationary points of thin elastic rods in higher-order scaling regimes (including the von Kármán scaling) under physical growth conditions for the elastic energy density has been established in [10].

In this paper we study the effects of simultaneous homogenization and dimension reduction in the context of convergence of stationary points in the von Kármán rod model. Let us denote by \( \Omega = (0, L) \times \omega \subset \mathbb{R}^3 \) a three-dimensional rod-like canonical domain of length \( L > 0 \) with cross-section \( \omega \subset \mathbb{R}^2 \) having a Lipschitz boundary. The (scaled) energy functional of a rod of thickness \( h > 0 \) occupying material domain \( \Omega_h = (0, L) \times h\omega \) associated to a deformation \( y^h : \Omega \to \mathbb{R}^3 \) is defined on the canonical domain by

\[
\mathcal{E}^h(y^h) = \int_{\Omega} W^h(x, \nabla_h y^h) dx - \int_{\Omega} f^h \cdot y^h dx.
\]

Above \( W^h \) is the elastic energy density describing an addmissible composite material (see Section 2.2), \( \nabla_h y^h = (\partial_1 y^h \mid \frac{1}{h} \partial_2 y^h \mid \frac{1}{h} \partial_3 y^h) \) denotes the scaled gradient of the deformation, and \( f^h \) describes an external load. It is well known that different scaling regimes with respect to the thickness \( h \) in the applied load and elastic energy lead at the limit to different rod models [13, 26]. In the von Kármán scaling of the rod, which is the subject of research here, we assume that the elastic energy of a minimizing sequence \( (y^h) \) satisfies

\[
\lim \sup_{h, i,0} \frac{1}{h^4} \int_{\Omega} W^h(x, \nabla_h y^h) dx < \infty,
\]

while the forcing term scales as \( f^h = h^3 f \) with \( f \in L^2((0, L), \mathbb{R}^3) \). Additionally, we assume inextensibility of the rod, i.e. \( f = f_2 e_2 + f_3 e_3 \) with \( f_2, f_3 \in L^2((0, L), \mathbb{R}) \), meaning that only normal loads to the mid-fiber of the rod are considered.

Under assumption (2) on a sequence of deformations \( (y^h) \) one can prove, based on the theorem of geometric rigidity [22], that there exist sequences of rotations \( (\hat{R}^h) \subset SO(3) \) and constants \( (c^h) \subset \mathbb{R}^3 \), such that transformed deformations \( \hat{y}^h = (\hat{R}^h)T y^h - c^h \) converge to the identity deformation on \( (0, L) \) in the \( L^2 \)-norm, i.e. \( \hat{y}^h \to x_1 e_1 \), and moreover, \( \nabla_h \hat{y}^h \to I \) in
the $L^2$-norm \cite{20} (cf. Theorem 2.1 below). Furthermore, scaled displacements, defined by

$$u^h(x_1) = \int_\omega \frac{\hat{y}_1^h - x_1}{h^2} \, dx', \quad v_i^h(x_1) = \int_\omega \frac{\hat{y}_i^h}{h} \, dx'$$

for $i = 2, 3$, and twist functions

$$w^h(x_1) = \frac{1}{\mu}\int_\omega \frac{x_2 \hat{y}_3^h - x_3 \hat{y}_2^h}{h^2} \, dx',$$

where $\mu(\omega) = \int_\omega (x_2^2 + x_3^2) \, dx'$, converge (weakly) on a suitably extracted subsequence to $(u, v_2, v_3, w) \in H^1(0, L) \times H^2(0, L) \times H^2(0, L) \times H^1(0, L)$ (see Theorem 2.2 for more details).

The strain sequence $(G^h)$ is implicitly defined through the decomposition of the scaled gradient as $\nabla \hat{y}^h = R^h(I + h^2G^h)$, where $(R^h)$ denotes the sequence of rotation functions constructed in Theorem 2.1. Convergence results from Theorems 2.1 and 2.2 allow for the representation of the symmetrized strain $\text{sym} G^h$ to the fixed and relaxation part as follows:

$$\text{sym} G^h = \text{sym}(\mu(m)) + \text{sym} \nabla \psi^h + o^h,$$

where the fixed part comes from

$$m = \begin{pmatrix} u' + \frac{1}{2} ((v_2')^2 + (v_3')^2) - v_2''x_2 - v_3''x_3 \\ -w'x_3 \\ w'x_2 \end{pmatrix},$$

the relaxation sequence $(\psi^h)$ satisfies $(\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0$, $\int_\omega (x_2 \psi_3^h - x_3 \psi_2^h) \, dx' \to 0$ in the $L^2$-norm and $\| \text{sym} \nabla \psi^h \|_{L^2(\Omega)} \leq C$, while the rest sequence $(o^h)$ converges to zero in the $L^2$-norm.

Using the $\Gamma$-convergence method accomplished for the bending rod model in \cite{19}, we can analogously perform the simultaneous homogenization and dimension reduction process under the assumption of the von Kármán scaling and inextensibility of the rod, and obtain the corresponding homogenized model whose energy is given by

$$\mathcal{E}^0(u, v_2, v_3, w) = K(h)(m) - \int_0^L (f_2v_2 + f_3v_3) \, dx_1,$$

where functions $u, v_2, v_3$ and $w$ are the weak limits of scaled displacements and twist function and $m$ as in (6). Moreover, the resulting limit elastic energy density (depending on a given subsequence of the diminishing thickness ($h$)) can be calculated according to

$$K(h)(m) = \lim_{h \to 0} \int_\omega Q^h(x, \mu(m) + \text{sym} \nabla \psi^h(m)) \, dx,$$

where $Q^h$ is the quadratic form approximating the energy density $W^h$, and $(\psi^h_m)$ the corresponding relaxation sequence. Confer Section 2.5 for more details.
The aim of this paper is to study the stationary points of the energy functional \( E_h \) rather than just global minimizers attainable through the \( \Gamma \)-convergence techniques. The Euler–Lagrange equation of the functional \( E_h \), assuming the von Kármán scaling and inextensibility, formally reads:

\[
\int_{\Omega} \left( DW_h(x, \nabla_h y^h) : \nabla_h \phi - h^3 (f_2 \phi_2 + f_3 \phi_3) \right) \, dx = 0 ,
\]

for all test functions \( \phi \in H^1_\omega(\Omega, \mathbb{R}^3) = \{ \phi \in H^1(\Omega) : \phi|_{\{0\} \times \omega} = 0 \} \). The latter notion of stationarity is the standard one, but possibly not best suited for the nonlinear elasticity. Alternative notion of stationarity in elasticity was proposed by Ball in [3], and that concept is compatible with a physical growth condition which roughly says that the energy blows up if the deformation degenerates. Stationarity condition (9) requires a technical assumption of a linear growth of the stress (cf. hypothesis H5 below), which is unphysical. While the authors in [10] managed to deal with the alternative stationarity condition and to systematically derive the corresponding stationarity conditions for the limit models, our method does not provide enough compactness to cope with the nonlinearities involved, and therefore, we reside in this setting.

Now we are in position to state the main result of the paper.

**Theorem 1.1.** Let the sequence \((W^h)\) describes an admissible composite material satisfying the von Kármán scaling (2) and let \( f^h = h^3 (f_2 \phi_2 + f_3 \phi_3) \) with \( f_2, f_3 \in L^2(0, L) \) be an external load. Let \((y^h)\) be a minimizing sequence for the energy functional \( E_h \) defined by (1), and assume that the transformed deformations \( \hat{y}^h \) are also stationary points of \( E_h \), i.e. solve equation (9), then

\[
\hat{y}^h \rightarrow x_1 e_1 \quad \text{in} \quad H^1(\Omega, \mathbb{R}^3) .
\]

Furthermore, the sequence of scaled displacements satisfy (on a subsequence)

\[
\begin{align*}
    u^h &\rightarrow u \quad \text{weakly in} \quad H^1(0, L) ; \\
    v_i^h &\rightarrow v_i \quad \text{strongly in} \quad H^1(0, L) , \quad \text{and} \quad v_i \in H^2(0, L) \quad \text{for} \quad i = 2, 3 ; \\
    w^h &\rightarrow w \quad \text{weakly in} \quad H^1(0, L) ,
\end{align*}
\]

where \((u, v_2, v_3, w)\) is a stationary point of the limit energy functional \( E^0 \) defined by (7).

Big part of the proof of Theorem 1.1 (compactness) is already available in the literature and these results are comprised and properly referenced in Theorems 2.1 and 2.2 below in Section 2. The novelty here is that the assumption of stationarity of the transformed deformations \( \hat{y}^h \) for the energy functional \( E_h \) (in the sense of (9)) implies the stationarity of the point \((u, v_2, v_3, w)\) for the limit energy functional \( E^0 \). The key point in proving that statement is the orthogonality property provided in Theorem 3.1 which essentially allows us to identify two relaxation sequences: \((\psi^h)\) from [5] and \((\psi^h_m)\) from [8], up to \( L^2 \)-concentrations which are irrelevant for identification of weak limits. The proof of Theorem 3.1 together with
the proof of Theorem 1.1 and identification of limit Euler-Lagrange equations are devised
to Section 3, while some technical results can be found in the appendix. We emphasize
at this point that, up to some technical peculiarities, the same approach can be utilized
for studying the convergence of stationary points of the von Kármán plate model, and the
analogous result holds true.

2. Preliminaries

2.1. Notation. \( \Omega = (0, L) \times \omega \subset \mathbb{R}^3 \) is a Lipschitz domain describing the canonical config-
uration of a rod of length \( L > 0 \) and shape \( \omega \subset \mathbb{R}^2 \). Vectors \( e_1, e_2, e_3 \) denote the canonical
basis of \( \mathbb{R}^3 \) and \( (x_1, x') \in \mathbb{R}^3 \), with \( x' = (x_2, x_3) \in \mathbb{R}^2 \), denote the coordinates of a point in
\( \mathbb{R}^3 \) with respect to that basis. Also, we will frequently use the projection of a point \( x \in \mathbb{R}^3 \)
to \( x' \)-plane, denoted by \( p_{x'}(x) = (0, x')^T \). For a given thickness \( h > 0 \), the scaled gradient is
denoted by \( \nabla_h = (\frac{1}{h} \partial_1, \frac{1}{h} \partial_2, \frac{1}{h} \partial_3) \). The space of real \( 3 \times 3 \) matrices is denoted by \( \mathbb{R}^{3 \times 3} \), while \( \mathbb{R}^{3 \times 3}_{\text{sym}}, \mathbb{R}^{3 \times 3}_{\text{skw}} \) and \( \text{SO}(3) \) denote the subspaces of symmetric, skew-symmetric, and orthogonal
matrices, respectively. For a skew-symmetric matrix \( A \) by \( \text{axl} A \) we denote its axial vector
by \( \text{axl} A = (A_{32}, A_{13}, A_{21}) \). Depending on the context, by \( |\cdot| \) we denote both the Lebesgue
measure of a set and the euclidean norm of a vector in \( \mathbb{R}^d \). The space of smooth functions
on \( (0, L) \) which are vanishing at zero will be denoted by \( \mathcal{C}_0^\infty(0, L) \). Given two functions
\( \phi, \psi \in L^1(\Omega, \mathbb{R}^3) \), we define its twist function \( t(\phi, \psi) : (0, L) \to \mathbb{R} \) by
\[
t(\phi, \psi)(x_1) = \int_\omega (x_2 \psi - x_3 \phi) dx'.
\]
Finally, the moments of a function \( \Psi \in L^1(\Omega, \mathbb{R}^{3 \times 3}) \) are denoted as follows. The zeroth
moment \( \Psi : (0, L) \to \mathbb{R}^{3 \times 3} \) defined by
\[
\Psi(x_1) = \int_\omega \Psi(x) dx', \tag{10}
\]
and first-order moments \( \tilde{\Psi}, \hat{\Psi} : (0, L) \to \mathbb{R}^{3 \times 3} \) defined by
\[
\tilde{\Psi}(x_1) = \int_\omega x_2 \Psi(x) dx', \quad \hat{\Psi}(x_1) = \int_\omega x_3 \Psi(x) dx'. \tag{11}
\]

2.2. von Kármán rod model – supplement. Let \( \omega \subset \mathbb{R}^2 \) be a Lipschitz domain of the
Lebesgue measure \( |\omega| = 1 \) and assume that coordinate axes are chosen such that
\[
\int_\omega x_2 dx' = \int_\omega x_3 dx' = \int_\omega x_2 x_3 dx' = 0.
\]
By \( \Omega^h = (0, L) \times h\omega \) we denote the reference configuration (material domain) of a rod-like
body of thickness \( h > 0 \) and length \( L > 0 \). Performing the standard change of variables
\( \Omega_h \ni \hat{x} \mapsto x \in \Omega \), given by \( x_1 = \hat{x}_1, \ x' = \frac{1}{h} \hat{x}' \), we will in the sequel work on the canonical
Let \( \Omega = (0, L) \times \omega \). For every \( h > 0 \), the (scaled) energy functional of a deformation \( y^h : \Omega \to \mathbb{R}^3 \) is given by expression (1).

For the elastic energy densities \( W^h \) we have more or less standard hypotheses on nonlinear composite material, which are listed in the sequel.

Nonlinear material law. Let \( \alpha, \beta, \varrho \) and \( \kappa \) be positive constants such that \( \alpha \leq \beta \). The class \( \mathcal{W}(\alpha, \beta, \varrho, \kappa) \) consists of all measurable functions \( W : \mathbb{R}^{3 \times 3} \to [0, +\infty] \) satisfying:

(1) frame indifference: \( W(RF) = W(F) \) for all \( F \in \mathbb{R}^{3 \times 3} \) and \( R \in SO(3) \);

(2) non-degeneracy:

\[
W(F) \geq \alpha \text{dist}^2(F, SO(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3},
\]

\[
W(F) \leq \beta \text{dist}^2(F, SO(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ with } \text{dist}^2(F, SO(3)) \leq \varrho;
\]

(3) minimality at identity: \( W(I) = 0 \);

(4) quadratic expansion at identity: \( W(I + G) = Q(G) + o(|G|^2) \) as \( G \to 0 \) \( (G \in \mathbb{R}^{3 \times 3}) \), where \( Q : \mathbb{R}^{3 \times 3} \to \mathbb{R} \) is a quadratic form;

(5) linear stress growth: \( |DW(F)| \leq \kappa(|F| + 1) \) for all \( F \in \mathbb{R}^{3 \times 3} \).

Admissible composite material. For \( \alpha, \beta, \varrho \) and \( \kappa \) positive constants as above, a family of functions \( W^h : \Omega \times \mathbb{R}^{3 \times 3} \to [0, +\infty] \) describes an admissible composite material of class \( \mathcal{W}(\alpha, \beta, \varrho, \kappa) \) if the following hypotheses hold:

(1) for every \( h > 0 \), \( W^h \) is almost everywhere equal to a Borel function on \( \Omega \times \mathbb{R}^{3 \times 3} \);

(2) for every \( h > 0 \), \( W^h(x, \cdot) \in \mathcal{W}(\alpha, \beta, \varrho, \kappa) \) for a.e. \( x \in \Omega \);

(3) there exists a monotone function \( r : \mathbb{R}_+ \to (0, +\infty) \) such that \( r(\delta) \downarrow 0 \) as \( \delta \downarrow 0 \) and

\[
\forall G \in \mathbb{R}^{3 \times 3}, \forall h > 0 : \text{ess sup}_{x \in \Omega} |W^h(x, I + G) - Q^h(x, G)| \leq r(|G|)|G|^2,
\]

where \( Q^h(x, \cdot) \) are quadratic forms defined in (4).

Given quadratic form \( Q^h(x, \cdot) \) can be (uniquely) represented by a positive semidefinite linear operator \( A^h(x) \),

\[
Q^h(x, F) = \frac{1}{2} A^h(x) F : F,
\]

for all \( F \in \mathbb{R}^{3 \times 3} \) and for a.e. \( x \in \Omega \).

Assuming that \( Q^h \) corresponds to an elastic energy density \( W^h \) belonging to a family of elastic energy densities describing an admissible composite material of class \( \mathcal{W}(\alpha, \beta, \varrho, \kappa) \), one can easily prove:

(a) \( |\text{sym } F|^2 \leq Q^h(x, F) = Q^h(x, \text{sym } F) \leq \beta|\text{sym } F|^2 \), for all \( F \in \mathbb{R}^{3 \times 3} \);

(b) \( |Q^h(x, F_1) - Q^h(x, F_2)| \leq \beta|\text{sym } F_1 - \text{sym } F_2| |\text{sym } F_1 + \text{sym } F_2| \), for all \( F_1, F_2 \in \mathbb{R}^{3 \times 3} \).
2.3. Rigidity and compactness. Using the theorem of geometric rigidity [12], the following result has been established in [20].

**Theorem 2.1.** Let \((y^h) \subset H^1(\Omega, \mathbb{R}^3)\) be a sequence satisfying
\[
\limsup_{h \downarrow 0} \frac{1}{h^4} \int_\Omega \text{dist}^2(\nabla_h y^h, \text{SO}(3))dx < +\infty.
\]

Then there exist: a sequence of maps \((R^h) \subset C^\infty([0,L], \text{SO}(3))\), a sequence of constant rotations \((\bar{R}^h) \subset \text{SO}(3))\) and constants \((c^h) \subset \mathbb{R}^3\) such that the sequence of deviations from the rigid motion \((\hat{y}^h)\), defined by \(\hat{y}^h = (\bar{R}^h)^T y^h - c^h\), satisfies
\[
\|\nabla_h \hat{y}^h - R^h\|_{L^2(\Omega)} \leq Ch^2, \tag{13}
\]
\[
\|(R^h)\|_{L^2(0,L)} \leq Ch, \tag{14}
\]
\[
\|R^h - I\|_{L^2(0,L)} \leq Ch. \tag{15}
\]

The sequence of constants \((c^h)\) in the previous theorem can be chosen such that
\[
\int_\Omega (\hat{y}^h_1 - x_1)dx = 0, \quad \int_\Omega \hat{y}^h_i dx = 0 \quad \text{for } i = 2, 3.
\]

Next, we introduce the following ansatz for \((\hat{y}^h)\):
\[
\hat{y}^h_1 = x_1 + h^2 \left( u^h - x_2 \frac{R^h_{21}}{h} - x_3 \frac{R^h_{31}}{h} \right) + h^2 \beta_1^h, \tag{16}
\]
\[
\hat{y}^h_i = h x_i + h v^h_i + h^2 w^h x_i^\perp + h^2 \beta_i^h, \quad \text{for } i = 2, 3,
\]
where \(x^\perp = (0, -x_3, x_2)\), and functions \(u^h, v^h_2, v^h_3, \) and \(w^h\) are defined in (3) and (4).

**Remark 2.1.** Observe that the proposed ansatz is a slight modification of the ansatz for the same sequence \((\hat{y}^h)\) from [20, Theorem 2.2 (f)]. In lieu of terms \((v^h_i)'\), \(i = 2, 3\), we set \(\frac{1}{h} R^h_{i1}\), respectively. This enables us to control the full scaled gradient of the corrector sequence \((\beta^h)\) in the \(L^2\)-norm (see Theorem 2.2 below), which is crucial for application of our method in the analysis afterwards.

**Theorem 2.2.** Let the assumption and notation of the previous theorem be retained. For sequences \((u^h), (v^h_i), i = 2, 3, \) and \((w^h)\) defined above, we have the following convergence results which hold on a subsequence:
\[
u^h_i \rightarrow v_i \text{ weakly in } H^1(0,L); \quad w^h \rightarrow w \text{ weakly in } H^1(0,L).
\]
Moreover, the sequence of corrector functions \((\beta^h)\) satisfies uniform bounds: \(\|\beta^h\|_{L^2(\Omega)} \leq Ch\) and \(\|\nabla_h \beta^h\|_{L^2(\Omega)} \leq C\).
Proof. The proof follows the lines of the proof of Theorem 2.2 from [20], but we include it here for the reader’s convenience. Let us define

$$A^h := \frac{1}{h} (R^h - I).$$

From the previous theorem we have $\|R^h - I\|_{L^2(0,L)} \leq Ch$ and $\|(R^h)^{\prime}\|_{L^2(0,L)} \leq Ch$, which implies the uniform bound $\|A^h\|_{H^1(0,L)} \leq C$. Therefore, (up to a subsequence) $A^h \rightharpoonup A$ weakly in $H^1((0,L),\mathbb{R}^{3 \times 3})$. From the compactness of the Sobolev embedding $H^1((0,L),\mathbb{R}^{3 \times 3}) \hookrightarrow L^\infty((0,L),\mathbb{R}^{3 \times 3})$, we conclude $A^h \to A$ strongly in $L^\infty((0,L),\mathbb{R}^{3 \times 3})$. Direct calculation reveals the following identities

$$A^h + (A^h)^T = -hA^h (A^h)^T \quad \text{and} \quad \frac{1}{h^2} \text{sym}(R^h - I) = \frac{1}{2h} (A^h + (A^h)^T),$$

which respectively imply $A^T = -A$ and

$$(17) \quad \frac{1}{h^2} \text{sym}(R^h - I) \to \frac{1}{2} A^2 \quad \text{strongly in } L^\infty((0,L),\mathbb{R}^{3 \times 3}).$$

Since $\|\nabla_h \hat{y}^h - R^h\|_{L^2(\Omega)} \leq Ch^2$, using the triangle inequality and established convergence results, we conclude

$$(18) \quad \frac{1}{h} (\nabla_h \hat{y}^h - I) \to A \quad \text{strongly in } L^2(\Omega,\mathbb{R}^{3 \times 3}).$$

By the construction $\int_0^L u^h(x_1)dx_1 = 0$. Thus, the Poincaré and Jensen inequalities together with (17) imply

$$\|u^h\|_{L^2(0,L)} \leq C_P \|((u^h)^{\prime})\|_{L^2(0,L)} \leq \frac{C_P}{h^2} \|\partial_1 \hat{y}_1^h - 1\|_{L^2(\Omega)}$$

$$\leq \frac{C_P}{h^2} \|\partial_1 \hat{y}_1^h - R^h_{11}\|_{L^2} + \frac{C_P}{h^2} \|R^h_{11} - 1\|_{L^2} \leq C.$$

Therefore, up to a subsequence $u^h \rightharpoonup u$ weakly in $H^1(0,L)$. Similarly, $\int_0^L v_i^h(x_1)dx_1 = 0$ for $i = 2,3$, and

$$\|v_i^h\|_{L^2(0,L)} \leq \frac{1}{h} \|\partial_1 \hat{y}_i^h\|_{L^2(\Omega)} \leq C.$$

Hence, (up to a subsequence) $v_i^h \rightharpoonup v_i$ weakly in $H^1(0,L)$. Moreover, since

$$(v_i^h)^{\prime} = \int_\omega \frac{\partial_1 \hat{y}_i^h}{h} dx \to A_{i1} \quad \text{strongly in } L^2(\Omega,\mathbb{R}^{3 \times 3}),$$

one concludes that $A_{i1} = v_i^\prime$ for $i = 2,3$. Since $A_{i1} \in H^1(0,L)$, we conclude $v_i \in H^2(0,L)$ for $i = 2,3$. Next, we consider the sequence of twist functions $(u^h)$. Note that they can be
written as
\[ w^h(x_1) = \frac{1}{\mu(\omega)} \int_\omega x_2 \left( \frac{h^{-1}y_3^h - x_3}{h} - \frac{1}{h^2} \int_\omega y_3^h \, dx' \right) \, dx' \]
\[ - \frac{1}{\mu(\omega)} \int_\omega x_3 \left( \frac{h^{-1}y_2^h - x_2}{h} - \frac{1}{h^2} \int_\omega y_2^h \, dx' \right) \, dx'. \]

For the above integrands we have (according to 18 and the Poincaré inequality):
\[ \frac{h^{-1}y_3^h - x_3}{h} - \frac{1}{h^2} \int_\omega y_3^h \, dx' \to A_{32} x_2 \quad \text{strongly in } L^2(\Omega); \]
\[ \frac{h^{-1}y_2^h - x_2}{h} - \frac{1}{h^2} \int_\omega y_2^h \, dx' \to -A_{32} x_3 \quad \text{strongly in } L^2(\Omega). \]

Therefore, \( w^h \) converges strongly in the \( L^2 \)-norm to the function \( w = A_{32} \in L^2(0, L) \). Using the a priori estimate \( \| \nabla h \hat{y}^h - R^h \|_{L^2} \leq Ch^2 \) and the normality of rotation matrix columns, we conclude the uniform bound \( \| (w^h)' \|_{L^2(0, L)} \leq C \). Hence, \( w^h \rightharpoonup w \) weakly in the \( H^1 \)-norm. Observe that the limit matrix \( A \in H^1((0, L), \mathbb{R}^{3 \times 3}) \) is completely identified by limits \( u, w \in H^1(0, L) \) and \( v_1, v_2 \in H^2(0, L) \) in the following way
\[
A = \begin{pmatrix} 0 & -v_2' & -v_3' \\ v_2' & 0 & -w \\ v_3' & w & 0 \end{pmatrix}.
\]

Finally, we consider the sequence of corrector functions \( (\beta^h) \) given by:
\[
\beta_1^h(x) = \frac{\hat{y}_1^h(x) - x_1}{h^2} - u^h(x_1) + x_2 \frac{R_{21}^h(x_1)}{h} + x_3 \frac{R_{31}^h(x_1)}{h};
\]
\[
\beta_i^h(x) = \frac{1}{h} \left( \frac{\hat{y}_i^h(x) - hx_i}{h} - v_i^h(x_1) - hw^h(x_1)x_i \right), \quad i = 2, 3.
\]

For brevity reasons, let us denote \( \partial_i^h = \frac{1}{h} \partial_i \), then for \( i = 2, 3 \) we compute
\[
\partial_i \beta_1^h = \frac{1}{h^2} \partial_i \hat{y}_1^h + \frac{R_{11}^h}{h} = \frac{1}{h} \left( \partial_i \hat{y}_1^h - R_{11}^h \right) + \frac{R_{11}^h + R_{1i}^h}{h}.
\]

The first term on the right is bounded in the \( L^2 \)-norm due to \( \| \nabla h \hat{y}^h - R^h \|_{L^2(\Omega)} \leq Ch^2 \), and the second one due to 17. Thus, \( \| \partial_i \beta_i^h \|_{L^2(\Omega)} \leq Ch \) for \( i = 2, 3 \). Since \( \int_\omega \beta_i^h(x) \, dx' = 0 \), using the Poincaré inequality we conclude
\[
\| \beta_1^h(x_1, \cdot) \|_{L^2(\omega)}^2 \leq C \left( \| \partial_2 \beta_1^h(x_1, \cdot) \|_{L^2(\omega)}^2 + \| \partial_3 \beta_1^h(x_1, \cdot) \|_{L^2(\omega)}^2 \right).
\]

Integrating the latter inequality along \( x_1 \)-direction yields the \( L^2(\Omega) \)-bound on \( \beta_1^h \) of order \( O(h) \). Identity
\[
\partial_1 \beta_1^h = \frac{\partial_1 \hat{y}_1^h - 1}{h^2} - (u^h)' + x_2 \frac{(R_{21}^h)'}{h} + x_3 \frac{(R_{31}^h)'}{h},
\]
directly implies the uniform bound \( \| \partial_1 \beta_i^h \|_{L^2(\Omega)} \leq C \). Straightforward calculations reveal

\[
\partial_1 \beta_i^h = \frac{1}{h} \left( \partial_1 \hat{y}_i^h - \delta_i - (-1)^j(1 - \delta_{ij})h w_i^h \right), \quad \text{for } i, j = 2, 3,
\]

where we have used \( \partial_1 x_i^h = (-1)^j(1 - \delta_{ij}) \). Furthermore,

\[
(sym \nabla \beta^h)_{ij} = \frac{\partial_1 \beta_i^h + \partial_1 \beta_j^h}{2} = \frac{1}{h} \left( sym(\nabla_h \hat{y}^h - I) \right)_{ij}, \quad \text{for } i, j = 2, 3,
\]

which implies the uniform bound \( \|(sym \nabla \beta^h)_{ij} \|_{L^2(\Omega)} \leq C \) for \( i, j = 2, 3 \). Note that for a.e. \( x_1 \in (0, L) \) the function \( (\beta_2^h(x_1, \cdot), \beta_3^h(x_1, \cdot)) \) belongs to the closed subspace \( \mathcal{B} = \left\{ \alpha \in H^1(\omega, \mathbb{R}^2) : \int_\omega \alpha(x') \, dx' = 0, \int_\omega (x_3 \alpha_2 - x_2 \alpha_3) \, dx' = 0 \right\} \), on which a Korn type inequality [25] holds

\[
\|\beta_2^h(x_1, \cdot)\|^2_{H^1(\omega)} + \|\beta_3^h(x_1, \cdot)\|^2_{H^1(\omega)} \leq C \sum_{i, j = 2, 3} \|(sym \nabla \beta^h)_{ij} \|^2_{L^2(\omega)}.
\]

Integrating the latter with respect to \( x_1 \), yields the respective uniform \( H^1(\Omega) \)-bound. Hence, we proved \( \|\beta^h\|_{L^2(\Omega)} \leq C h \). Finally,

\[
\partial_1 \beta_i^h = \frac{1}{h} \left( \frac{\partial_1 \hat{y}_i^h}{h} - (v_i^h)' - h(w_i^h)'x_i^h \right)
= \frac{1}{h} \left( \frac{\partial_1 \hat{y}_i^h - R_i^h}{h} - \frac{1}{h} \int_\omega (\partial_1 \hat{y}_i^h - R_i^h) \, dx' - h(w_i^h)'x_i^h \right), \quad \text{for } i, j = 2, 3,
\]

and the previously established convergence results imply \( \|\partial_1 \beta_i^h\|_{L^2(\Omega)} \leq C \). Thus, we have proved \( \|\nabla_h \beta^h\|_{L^2(\Omega)} \leq C \). \qed

2.4. Strain and stress estimates. For every \( h > 0 \), using the rotation matrix function \( R^h \), the strain tensor \( G^h \) is implicitly defined through the following decomposition of the scaled deformation gradient

\[
\nabla_h \hat{y}^h = R^h (I + h^2 G^h).
\]

The explicit identity \( G^h = h^{-2}(R^h)^T(\nabla_h \hat{y}^h - R^h) \) directly with (13) implies the \( L^2 \)-uniform bound on the sequence \( (G^h) \). Hence, there exists \( G \in L^2(\Omega, \mathbb{R}^{3 \times 3}) \) such that \( G^h \rightharpoonup G \) on a subsequence. Aim is to describe the symmetrized strain \( \text{sym} G^h \) more in detail. First, we explicitly involve limit functions \( u, w \in H^1(0, L) \) and \( v_1, v_2 \in H^2(0, L) \) into our ansatz (16) in the following way:

\[
\frac{\hat{y}_1^h - x_1}{h^2} = u - x_2 v_2' - x_3 v_3' + \psi_1^h, \quad \frac{\hat{y}_i^h - h x_i}{h^2} = \frac{v_i}{h} + w x_i^h + \psi_i^h, \quad \text{for } i = 2, 3,
\]
where
\[
\psi^h_i = u^h - u - x_2\left(\frac{P^h_{21}}{h} - v_2'\right) - x_3\left(\frac{P^h_{31}}{h} - v_3'\right) + \beta^h_i ,
\]
\[
\psi^h_i = \frac{1}{h}(v_i^h - v_i) + (w^h - w)x_i^h + \beta^h_i , \quad \text{for } i = 2, 3 .
\]

Previously established convergence results imply that \((\psi^h_1, h\psi^h_2, h\psi^h_3) \to 0\) strongly in the \(L^2\)-norm. Moreover, performing straightforward calculations

\[
\partial_1\psi^h_i = (u^h)' - u' - x_2\left(\frac{(P^h_{21})'}{h} - v_2''\right) - x_3\left(\frac{(P^h_{31})'}{h} - v_3''\right) + \partial_1\beta^h_i,
\]

\[
\partial_j^h\psi^h_i = \frac{v_j'}{h} - \frac{R^h_{j1}}{h^2} + \partial_j^h\beta^h_i , \quad \text{for } j = 2, 3 ,
\]

\[
\partial_j^h\psi^h_i = \frac{(-1)^j}{h}(1 - \delta_{ij})(u^h - w) + \partial_j^h\beta^h_i , \quad \text{for } i, j = 2, 3 ,
\]

\[
\partial_1\psi^h_i = \frac{1}{h} \left( (v^h_i)'' - v_i'' \right) + \left( (w^h)' - w' \right) x_i^h + \partial_1\beta^h_i , \quad \text{for } i = 2, 3 ,
\]

and known convergence results, immediately gives \(\| \text{sym } \nabla_h \psi^h \|_{L^2(\Omega)} \leq C\). Invoking (19), we obtain the following representation:

\[
(20) \quad \frac{1}{h^2} \text{sym} \left( \nabla_h \psi^h \right) - I = u'e_1 \otimes e_1 + \text{sym}(\nabla(A'p_{x'})) + \text{sym} \nabla_h \psi^h ,
\]

where \(p_{x'}(x) = (0, x')^T\) denotes the projection of \(x \in \mathbb{R}^3\) on \(x'-\text{plane}\). Additionally, using \((\beta^h_2(x_1, \cdot), \beta^h_3(x_1, \cdot)) \in \mathcal{B}\) for a.e. \(x_1 \in (0, L)\), one can easily check that

\[
\int_\omega (x_3^h\psi^h_2 - x_2^h\psi^h_3)dx' = -(w^h - w) \int_\omega (x_2^2 + x_3^2)dx' \to 0 \quad \text{strongly in } L^2 .
\]

Next, we compute the symmetrized strain using decomposition (20):

\[
\text{sym } G^h = \frac{1}{h^2} \left( R^h \right)^T (\nabla_h y^h - I) = \frac{1}{h^2} \text{sym}((R^h - I)^T \nabla_h y^h) + \frac{1}{h^2} \text{sym}(\nabla_h y^h - I)
\]

\[
= \frac{1}{h^2} \text{sym}((R^h - I)^T(\nabla_h y^h - R^h)) - \frac{1}{h^2} \text{sym}(R^h - I) + \frac{1}{h^2} \text{sym}(\nabla_h y^h - I)
\]

\[
= \partial^h - \frac{1}{h^2} \text{sym}(R^h - I) + u'e_1 \otimes e_1 + \text{sym}(\nabla(A'p_{x'})) + \text{sym} \nabla_h \psi^h
\]

\[
= \partial^h + \text{sym} \nabla_h \psi^h + o^h ,
\]

where \(o^h \to 0\) strongly in \(L^2(\Omega, \mathbb{R}^{3 \times 3})\), and \(\text{sym } H = u'e_1 \otimes e_1 + \text{sym}(\nabla(A'p_{x'})) - \frac{1}{2} A^2\). In this way we decomposed \(\text{sym } G^h\) into a fixed and relaxation part. A part of \(\text{sym } H\) can be
further transferred to the relaxation terms as follows
\[
\text{sym } H = \left( u' + \frac{1}{2} \left( (v'_2)^2 + (v'_3)^2 \right) \right) e_1 \otimes e_1 \quad \text{sym}(\mu(A'p'_x)) \\
+ \frac{1}{2} \left( \begin{array}{ccc}
0 & v'_3w & -v'_2w \\
v'_3w & w^2 + (v'_2)^2 & v'_2v'_3 \\
-v'_2w & v'_2v'_3 & w^2 + (v'_3)^2
\end{array} \right)
\]
\[= \text{sym}(\mu(m)) + \text{sym} \nabla_h \alpha^h - \text{sym} \mu(\partial_1 \alpha^h),\]
where
\[
(21) \quad m = \left( u' + \frac{1}{2} \left( (v'_2)^2 + (v'_3)^2 \right) \right) e_1 + A'p'_x,
\]
and
\[
\alpha^h(x) = h \left( \begin{array}{c}
x_2v'_3w - x_3v'_2w \\
\frac{1}{2}x_2(w^2 + (v'_2)^2) + \frac{1}{2}x_3v'_2v'_3 \\
\frac{1}{2}x_2v_2v'_3 + \frac{1}{2}x_3(w^2 + (v'_3)^2)
\end{array} \right).
\]
Finally, we have decomposition
\[
(22) \quad \text{sym } G^h = \text{sym}(\mu(m)) + \text{sym} \nabla_h \psi^h + o^h,
\]
with updated relaxation and \(L^2\)–zero convergent parts.

The stress field \(E^h : \Omega \to \mathbb{R}^{3 \times 3}\) is defined by
\[
E^h := \frac{1}{h^2} DW^h(\cdot, I + h^2 G^h).
\]
Using the assumption (C3) on \(W^h\), in particular estimate \([12]\), implies that \(W^h\) is differentiable \(a.e.\) in \(x \in \Omega\) and
\[
(23) \quad \forall G \in \mathbb{R}^{3 \times 3}, \forall h > 0 : \text{ess sup}_{x \in \Omega} |DW^h(x, I + G) - A^h(x)G| \leq r(|G|)|G|,
\]
and therefore,
\[
|DW^h(\cdot, I + h^2 G^h)| \leq r(h^2|G^h|)h^2|G^h| + \beta h^2|G^h| \quad \text{a.e. in } \Omega.
\]
Let us denote the set
\[
B_h := \{ x \in \Omega : h^2|G^h(x)| \leq 1 \},
\]
then from the previous inequality
\[
|DW^h(\cdot, I + h^2 G^h)| \leq C h^2|G^h| \quad \text{pointwise in } B_h,
\]
which yields
\[
|E^h| \leq C|G^h| \quad \text{pointwise in } B_h.
\]
On the other hand on \(\Omega \setminus B_h\), i.e. on the set where \(|G^h| > h^{-2}\) \(a.e.\), applying hypothesis (H5) we conclude
\[
|E^h| \leq \frac{K}{h^2} (|I + h^2 G^h| + 1) \leq \frac{K}{h^2} \left( h^2|G^h| + \sqrt{3} + 1 \right) \leq C|G^h| \quad \text{pointwise in } \Omega \setminus B_h.
\]
Therefore, we have a uniform estimate on the whole set,
\begin{equation}
|E^h| \leq C|G^h| \quad \text{pointwise in } \Omega,
\end{equation}
which together with the uniform $L^2$-bound for the strain sequence $(G^h)$ implies the uniform $L^2$-bound on $(E^h)$ and consequently the weak convergence (on a subsequence)
\begin{equation}
E^h \rightharpoonup E \quad \text{in } L^2(\Omega, \mathbb{R}^{3\times3}).
\end{equation}

2.5. **Representation of elastic energy functionals.** In this subsection we briefly recall a variational approach for general (non-periodic) simultaneous homogenization and dimension reduction in the framework of three-dimensional nonlinear elasticity theory. This approach has been thoroughly undertaken in case of von Kármán plate [27] and bending rod [19], while the linear plate model has been outlined in [7]. The theorem on geometric rigidity provides a decomposition of the symmetrized strain to a sum of a fixed and relaxation part (cf. previous section). Utilizing corresponding Griso’s decomposition [15, 16] gives a further characterization of the relaxation part, which enables an operational representation of elastic energies (cf. Lemma 2.3 below), suitable for the application of appropriate $\Gamma$-convergence techniques to eventually identify the limiting elastic energy.

In the following we only provide basic steps of the method and state the final results. To start with, let us define so called lower and upper $\Gamma$-limits. For a monotonically decreasing to zero sequence of positive numbers $(h) \subset (0, +\infty)$, $m \in L^2(\Omega, \mathbb{R}^3)$ and open set $O \subset (0, L)$, we define:
\begin{equation}
K_{(h)}^-(m, O) = \inf \left\{ \liminf_{h \downarrow 0} \int_{O \times \omega} Q^h(x, \text{sym } m + \text{sym } \nabla_h \psi^h) \, dx \mid \psi_1^h, \psi_2^h, \psi_3^h \to 0 \text{ in } L^2(O \times \omega, \mathbb{R}^3), \ t(\psi_2^h, \psi_3^h) \to 0 \text{ in } L^2(O) \right\};
\end{equation}
\begin{equation}
K_{(h)}^+(m, O) = \inf \left\{ \limsup_{h \downarrow 0} \int_{O \times \omega} Q^h(x, \text{sym } m + \text{sym } \nabla_h \psi^h) \, dx \mid \psi_1^h, \psi_2^h, \psi_3^h \to 0 \text{ in } L^2(O \times \omega, \mathbb{R}^3), \ t(\psi_2^h, \psi_3^h) \to 0 \text{ in } L^2(O) \right\};
\end{equation}
The above infimization is taken over all sequences $(\psi^h) \subset H^1(O \times \omega, \mathbb{R}^3)$ such that $(\psi_1^h, \psi_2^h, \psi_3^h) \to 0$ and twist functions $t(\psi_2^h, \psi_3^h) \to 0$ strongly in the $L^2$-topology as $h \to 0$. Identical proof to the one presented for Lemma 3.4 in [27] gives the continuity of $K_{(h)}^-$ and $K_{(h)}^+$ with respect to the first variable. Utilizing the diagonal procedure yields the equality of $K_{(h)}^-$ and $K_{(h)}^+$ for a subsequence, still denoted by $(h)$, on $L^2(\Omega, \mathbb{R}^3) \times \mathcal{O}$, where $\mathcal{O}$ denotes a countable family of open subsets of $(0, L)$. This asserts the definition of the functional
\begin{equation}
K_{(h)}(m, O) := K_{(h)}^-(m, O) = K_{(h)}^+(m, O), \quad \forall m \in L^2(\Omega, \mathbb{R}^3), \quad \forall O \in \mathcal{O}.
\end{equation}
Lemma 2.3. Let \((h) \subset (0, +\infty), h \downarrow 0\), be a sequence of positive numbers which satisfies (26) for every open set \(O \subset (0, L)\). Then there exists a subsequence, still denoted by \((h)\), which satisfies that for every \(m \in L^2(\Omega, \mathbb{R}^3)\) there exists \((\psi^h) \subset H^1(\Omega, \mathbb{R}^3)\) such that for every open subset \(O \subset (0, L)\), we have

\[
K_{(h)}(m, O) = \lim_{h \downarrow 0} \int_{O \times \omega} Q^h(x, \text{sym } \iota(m) + \text{sym } \nabla_h \psi^h)dx,
\]

and the following properties hold:

(a) \((\psi^h_1, h\psi^h_2, h_n \psi^h_3) \to 0\) and \(t(\psi^h_2, \psi^h_3) \to 0\) strongly in the \(L^2\)-norm as \(h \downarrow 0\),

(b) sequence \((\text{sym } \nabla_h \psi^h)^2\) is equi-integrable and there exist sequences \((\Psi^h) \subset H^1((0, L), \mathbb{R}^{3 \times 3}_{\text{skw}})\) and \((\vartheta^h) \subset H^1(\Omega, \mathbb{R}^3)\) satisfying: \(\Psi^h \to 0\), \(\vartheta^h \to 0\) strongly in the \(L^2\)-norm, and

\[
\text{sym } \nabla_h \psi^h = \iota((\Psi^h)^p_{x_r}) + \text{sym } \nabla_h \vartheta^h.
\]

Moreover, \((|\Psi^h|^2)\) and \((|\nabla_h \vartheta^h|^2)\) are equi-integrable and the following inequality holds

\[
\limsup_{h \downarrow 0} \left( \| \Psi^h \|_{H^1(O)} + \| \nabla_h \vartheta^h \|_{L^2(O \times \omega)} \right) \leq C(\|m\|_{L^2(\Omega \times \omega)} + 1),
\]

for some \(C > 0\) independent of \(O \subset (0, L)\).

(c) (orthogonality) if \((\varphi^h) \subset H^1(\Omega, \mathbb{R}^3)\) is any other sequence that satisfies (a) and \((\text{sym } \nabla_h \varphi^h)\) is bounded in \(L^2(\Omega, \mathbb{R}^{3 \times 3})\), then

\[
\lim_{h \downarrow 0} \int_{\Omega} (\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \text{sym } \nabla_h \varphi^h dx = 0;
\]

(d) (uniqueness) if \((\varphi^h) \subset H^1(\Omega, \mathbb{R}^3)\) is any other sequence that satisfies (27) and (a), then

\[
\| \text{sym } \nabla_h \psi^h - \text{sym } \nabla_h \varphi^h \|_{L^2(\Omega)} \to 0,
\]

and \((|\nabla_h \varphi^h|^2)\) is equi-integrable.

An important feature of the method is the localization property of the relaxation sequence, i.e. if we know the relaxation sequence for the interval \((0, L)\), the relaxation sequence for an arbitrary open subset \(O \subset (0, L)\) and fixed \(m \in L^2(\Omega, \mathbb{R}^3)\), is simply obtained by restriction.

Finally, we provide the integral representation of the functional \(K_{(h)}\) (cf. [19, Proposition 2.12]). Recall from (21) that \(m\) is of the form \(m = (u' + \frac{1}{2}((v_2')^2 + (v_3')^2))e_1 + A'p_{x'}\). Therefore, we consider the mapping \(m : L^2((0, L), \mathbb{R}^{3 \times 3}_{\text{skw}}) \times L^2(0, L) \to L^2(\Omega, \mathbb{R}^3)\) defined by \(m(\varrho, \Psi) = \varrho e_1 + \Psi_{x'}\).

Proposition 2.4. Let \((h) \subset (0, +\infty)\) be a sequence monotonically decreasing to zero. Then there exists a subsequence still denoted by \((h)\) and a measurable function \(Q^h : (0, L) \times \)
\[ \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \text{ depending on } (h), \text{ such that for every open subset } O \subset (0, L) \text{ and every } (\varrho, \Psi) \in L^2(0, L) \times L^2((0, L), \mathbb{R}^{3 \times 3}_{skw}) \]

(29) \[ K(h)(m(\varrho, \Psi), O) = \int_O Q^0(x_1, \varrho(x_1), axl \Psi(x_1)) \, dx_1. \]

Moreover, for a.e. \( x_1 \in (0, L) \), \( Q^0(x_1, \cdot, \cdot) : \mathbb{R}^4 \to \mathbb{R} \) is a bounded and coercive quadratic form.

At this point we also define function \( Q^0_1 : (0, L) \times \mathbb{R}^3 \to \mathbb{R} \) by

\[ Q^0_1(x_1, F) = \min_{z \in \mathbb{R}} Q^0(x_1, z, F) \quad \text{for all } F \in \mathbb{R}^3 \text{ and a.e. } x_1 \in (0, L), \]

and function \( \varrho_0 : (0, L) \times \mathbb{R}^3 \to \mathbb{R} \) satisfying \( Q^0_1(x_1, axl F) = Q^0(x_1, \varrho_0(x_1, axl F), axl F) \) for all \( F \in \mathbb{R}^{3 \times 3}_{skw} \) and a.e. \( x_1 \in (0, L) \). Linear operators associated with quadratic forms \( Q^0_1(x_1, \cdot, \cdot) \) and \( Q^0_1(x_1, \cdot) \) are denoted by \( A^0_1(x_1) \) and \( A^1_1(x_1) \), respectively.

2.6. Variational derivative of the limit elastic energy. Let \((h) \subset (0, +\infty) \) be a monotonically to zero decreasing sequence of positive numbers and let \( m \in L^2(\Omega, \mathbb{R}^3) \) be given. According to Lemma 2.3, there exist a subsequence still denoted by \((h)\) and a relaxation sequence \((\psi^h_m) \subset H^1(\Omega, \mathbb{R}^3)\) depending on \( m \) and satisfying \( (\psi^h_{m,1}, h\psi^h_{m,2}, h\psi^h_{m,3}) \to 0 \) and \( t(\psi^h_{m,2}, \psi^h_{m,3}) \to 0 \) strongly in the \( L^2 \)-norm, such that the limit elastic energy \( K(h)(m) := K_{(h)}(m, (0, L)) \) is given by

\[ K_{(h)}(m) = \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h(m) : \text{sym} \, \nabla_h \psi^h_m \, dx \]

In the following we compute the variational derivative of \( K(h) \) at point \( m \in L^2(\Omega, \mathbb{R}^3) \). Let \( n \in C^\infty(\Omega, \mathbb{R}^3) \), such that \( n(0, x') = 0 \) for all \( x' \in \Omega \), be a test function. Then by the definition, variational derivative is given as

(30) \[ \frac{\delta K(m)}{\delta m}[n] = \lim_{\varepsilon \downarrow 0} \frac{K(m + \varepsilon n) - K(m)}{\varepsilon}. \]
With a trick of successive adding of the corresponding relaxation sequences and using the orthogonality property \( (28) \), for a suitable subsequence of \((h)\) we calculate:

\[
\mathcal{K}(h)(m + \varepsilon n) - \mathcal{K}(h)(m) = \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m + \varepsilon n) + \nabla_h \psi^h_{m+\varepsilon n}) : (\text{sym} \ \iota(m + \varepsilon n) + \nabla_h \psi^h_{m+\varepsilon n}) \ dx
\]

\[
- \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) : (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) \ dx
\]

\[
= \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m + \varepsilon n) + \nabla_h \psi^h_{m+\varepsilon n}) : \text{sym} \ \iota(m + \varepsilon n) \ dx
\]

\[
- \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) : \text{sym} \ \iota(m) \ dx
\]

\[
= \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m + \varepsilon n) + \nabla_h \psi^h_{m+\varepsilon n}) : (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) \ dx
\]

\[
+ \lim_{h \downarrow 0} \frac{\varepsilon}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m + \varepsilon n) + \nabla_h \psi^h_{m+\varepsilon n}) : (\text{sym} \ \iota(n) + \nabla_h \psi^h_n) \ dx
\]

\[
- \lim_{h \downarrow 0} \frac{1}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) : \text{sym} \ \iota(n) \ dx
\]

\[
= \lim_{h \downarrow 0} \frac{\varepsilon}{2} \int_{\Omega} A^h (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) : \text{sym} \ \iota(n) \ dx
\]

\[
+ \lim_{h \downarrow 0} \frac{\varepsilon^2}{2} \int_{\Omega} A^h (\text{sym} \ \iota(n) + \nabla_h \psi^h_n) : \text{sym} \ \iota(n) \ dx
\]

Finally, according to the definition \((30)\) and utilizing the uniform \(L^\infty\)-bound for the sequence of tensors \((A^h)\), we infer

\[
\frac{\delta \mathcal{K}(h)(m)}{\delta m}[n] = \lim_{h \downarrow 0} \int_{\Omega} A^h (\text{sym} \ \iota(m) + \nabla_h \psi^h_m) : \text{sym} \ \iota(n) \ dx .
\]
3. Derivation of homogenized Euler-Lagrange equations — proof of Theorem 1.1

Taking the \( L^2 \)-variation of the energy functional \( \mathcal{E}^h \) defined by (1), one finds formally the Euler–Lagrange equation:

\[
(32) \quad \frac{\delta \mathcal{E}^h(y^h)}{\delta y^h} \phi = \int_\Omega \left( D W^h(x, \nabla_h y^h) : \nabla_h \phi - h^3 (f_2 \phi_2 + f_3 \phi_3) \right) \, dx = 0,
\]

for all test functions \( \phi \in H^1_0(\Omega, \mathbb{R}^3) \). Let \( y^h \) be a stationary point of \( \mathcal{E}^h \), i.e. it satisfies (32). From the frame indifference of \( W^h \) it follows that \( R^T D W^h(x, RF) = D W^h(x, F) \) for all \( R \in \text{SO}(3), F \in \mathbb{R}^{3 \times 3} \) and a.e. \( x \in \Omega \), which implies (using that \( \nabla_h y^h = R^h(I + h^2 G^h) \))

\[
(33) \quad D W^h(x, \nabla_h y^h) = R^h D W^h(x, I + h^2 G^h) = h^2 R^h E^h.
\]

Taylor expansion around the identity gives

\[
D W^h(x, I + h^2 G^h) = h^2 G^h + \zeta^h(x, h^2 G^h),
\]

where \( \zeta^h \) is such that \( |\zeta^h(\cdot, F)|/|F| \leq r(|F|) \) uniformly in \( \Omega \), for all \( F \in \mathbb{R}^{3 \times 3} \) and \( h > 0 \). The latter follows from the assumption (12) on admissible composite materials. Since \( D^2 W^h(x, I) = A^h(x) \) and \( A^h(x) \) is symmetric tensor, the above identity yields

\[
(34) \quad E^h(x) = A^h(x) \text{sym} \ G^h(x) + \frac{1}{h^2} \zeta^h(x, h^2 G^h),
\]

which after employing (22) leads to

\[
(35) \quad E^h = A^h(\text{sym} \ i(m)) + \text{sym} \nabla_h \psi^h + \frac{1}{h^2} \zeta^h(\cdot, h^2 G^h) + A^h \phi^h.
\]

3.1. Orthogonality property.

Theorem 3.1. Let \( (A^h) \) be a sequence of tensors describing an admissible composite material, let \( m \) be the fixed part of the symmetrized strain defined by (21), and \( (\psi^h) \subset H^1(\Omega, \mathbb{R}^3) \) the corresponding relaxation sequence satisfying \( (\psi^h_1, h \psi^h_2, h \psi^h_3) \to 0 \), \( (\psi^h_1, \psi^h_2, \psi^h_3) \to 0 \) strongly in the \( L^2 \)-norm and \( \| \text{sym} \nabla_h \psi^h \|_{L^2(\Omega)} \leq C \). Then, for every sequence \( (\varphi^h) \subset H^1(\Omega, \mathbb{R}^3) \) satisfying \( (\varphi^h_1, h \varphi^h_2, h \varphi^h_3) \to 0 \), \( (\varphi^h_2, \varphi^h_3) \to 0 \) strongly in the \( L^2 \)-norm and \( (\| \text{sym} \nabla_h \varphi^h \|^2) \) is equi-integrable, the following orthogonality property holds

\[
(36) \quad \lim_{h \to 0} \int_\Omega A^h(\text{sym} \ i(m)) + \text{sym} \nabla_h \psi^h : \text{sym} \nabla_h \varphi^h \, dx = 0.
\]

Proof. Let \( (\psi^h) \subset H^1(\Omega, \mathbb{R}^3) \) and \( (\varphi^h) \subset H^1(\Omega, \mathbb{R}^3) \) be arbitrary sequences satisfying the assumptions of the theorem. Applying the Griso’s decomposition for the sequence \( (\varphi^h) \) (cf. [19, Corollary 2.3]), there exist sequences \( (\Phi^h) \subset H^1((0, L), \mathbb{R}^{3 \times 3}_{sym}), (\phi^h) \subset H^1(\Omega, \mathbb{R}^3) \) and \( (\delta^h) \subset L^2(\Omega, \mathbb{R}^{3 \times 3}) \) satisfying:

\[
\text{sym} \nabla_h \varphi^h = \text{sym} \ i((\Phi^h)' \ p_{x'}) + \text{sym} \nabla_h \phi^h + \delta^h,
\]
\( \Phi^h \to 0, \phi^h \to 0, \rho^h \to 0 \) strongly in the \( L^2 \)-norm, and

\[
\| \Phi^h \|_{H^1(0,L)} + \| \phi^h \|_{L^2(\Omega)} + \| \nabla h \phi^h \|_{L^2(\Omega)} \leq C \| \nabla h \varphi^h \|_{L^2(\Omega)}, \quad \forall h > 0.
\]

Furthermore, there exist subsequences \((\Phi^h)\) and \((\phi^h)\) (still denoted by \((h)\)) and sequences \((\tilde{\Phi}^h) \subset H^1((0,L),\mathbb{R}^3)\) and \((\tilde{\phi}^h) \subset H^1(\Omega,\mathbb{R}^3)\) such that \( \{\Phi^h \neq \tilde{\Phi}^h\} \cup \{\Phi^h \neq \tilde{\Phi}^h\}' \neq 0 \) and \( \{\phi^h \neq \tilde{\phi}^h\} \cup \{\nabla \phi^h \neq \nabla \tilde{\phi}^h\} \to 0 \) as \( h \downarrow 0 \), and sequences \((|\Phi^h|^2)\) and \((|\nabla h \phi^h|^2)\) are equi-integrable (cf. [14] and [19] Lemma 2.17). The rest of the proof will be divided into two parts showing the property \((\text{36})\) for sequences \((\tilde{\phi}^h)\) and \((\tilde{\Phi}^h)\), respectively. For ease of presentation, we will in further denote these sequences again by \((\phi^h)\) and \((\Phi^h)\).

**Part 1.** The equi-integrability property of the sequence \((\phi^h)\) allows us to modify each \(\phi^h\) to zero near the boundary (cf. [27] Lemma 3.6), thus, making it an eligible test function in the Euler–Lagrange equation \((32)\). Using the identity \((35)\) and the modified \(\phi^h\) as a test function in the Euler–Lagrange equation \((32)\), we have (according to \((33)\))

\[
\int_{\Omega} R^h \mathbb{A}^h (\text{sym } t(m) + \text{sym } \nabla h \psi^h) : \nabla h \phi^h dx = \int_{\Omega} R^h \left( E^h - \frac{1}{h^2} \zeta^h(\cdot,h^2 G^h) - \mathbb{A}^h o^h \right) : \nabla h \phi^h dx
\]

\[
= \int_{\Omega} h (f_2 \phi^h_2 + f_3 \phi^h_3) - \int_{\Omega} R^h \left( \frac{1}{h^2} \zeta^h(\cdot,h^2 G^h) + \mathbb{A}^h o^h \right) : \nabla h \phi^h dx.
\]

Obviously, the first and the last term converge to 0 as \( h \downarrow 0 \). Let us examine the second term

\[
\frac{1}{h^2} \int_{\Omega} R^h \zeta^h(\cdot,h^2 G^h) : \nabla h \phi^h dx.
\]

Denote the set \( S^\alpha_h := \{ x \in \Omega : h^2 |G^h(x)| \leq h^\alpha \} \) for some \( 0 < \alpha < 2 \). On \( S^\alpha_h \) we have

\[
|\zeta^h(\cdot,h^2 G^h)|_{G^h} \leq \sup \left\{ \frac{|\zeta^h(\cdot,h^2 \tilde{G}^h)|}{h^2 |\tilde{G}^h|} : h^2 |\tilde{G}^h| \leq h^\alpha \right\} |\tilde{G}^h| \leq r(h^\alpha)|G^h|.
\]

Therefore,

\[
\frac{1}{h^2} \int_{S^\alpha_h} R^h \zeta^h(\cdot,h^2 G^h) : \nabla h \phi^h dx \leq r(h^\alpha) R^h \|G^h\|_{L^2(\Omega)} \|G^h\|_{L^2(\Omega)} \|\nabla h \phi^h\|_{L^2(\Omega)} \leq C r(h^\alpha) \to 0,
\]

as \( h \downarrow 0 \). On the other hand, on \( \Omega \setminus S^\alpha_h \) we have a pointwise a.e. bound

\[
\frac{1}{h^2} |\zeta^h(\cdot,h^2 G^h)| \leq C |G^h| \quad \text{a.e. on } \Omega \setminus S^\alpha_h,
\]

which in fact holds pointwise a.e. on \( \Omega \). This follows by the triangle inequality from \((34)\) using \((24)\) and \(|\mathbb{A}^h(x)G^h(x)| \leq \beta |G^h(x)|\) for a.e. \( x \in \Omega \). Therefore, using the Hölder and
Chebyshev inequalities, respectively, we find
\[
\frac{1}{h^2} \left| \int_{\Omega \setminus S_h^a} R^h \zeta^h(\cdot, h^2 G^h) : \nabla_h \phi^h \, dx \right| \leq C \int_{\Omega \setminus S_h^a} |G^h| |\nabla_h \phi^h| \, dx
\]
\[
\leq C \|\nabla_h \phi^h\|_{L^\infty(\Omega \setminus S_h^a)} \int_{\Omega \setminus S_h^a} |G^h| \, dx
\]
\[
\leq C \|\nabla_h \phi^h\|_{L^\infty(\Omega)} |\Omega \setminus S_h^a|^{1/2}
\]
\[
\leq C \|\nabla_h \phi^h\|_{L^\infty(\Omega)} h^{1 - \alpha/2}.
\]

In order to successfully pass to the limit when \( h \downarrow 0 \), we have to again replace the sequence \((\phi^h)\) by a sequence obtained by means of Corollary A.2 (cf. Lemma A.1). We choose a strictly increasing sequence \((s_h) \subset (0, +\infty)\) defined by \( s_h = h^{-\gamma} \) with a constant \( \gamma > 0 \) satisfying \( 1 - \alpha/2 - \gamma > 0 \). The obtained sequence \((\tilde{\phi}^h)\) then satisfies \( \|\nabla_h \tilde{\phi}^h\|_{L^\infty(\Omega)} \leq C s_h \), and we infer
\[
\lim_{h \downarrow 0} \frac{1}{h^2} \left| \int_{\Omega \setminus S_h^a} R^h \zeta^h(\cdot, h^2 G^h) : \nabla_h \tilde{\phi}^h \, dx \right| = 0.
\]

Thus, we have shown
\[
\lim_{h \downarrow 0} \int_{\Omega} R^h A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \nabla_h \tilde{\phi}^h \, dx = 0.
\]

Since \( R^h \to I \) strongly in the \( L^\infty \)-norm, it follows that
\[
\lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \nabla_h \tilde{\phi}^h \, dx = 0,
\]
while the symmetry property of \( A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) \) eventually implies
\[
(38) \quad \lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \text{sym } \nabla_h \tilde{\phi}^h \, dx = 0.
\]

Due to the fact that \( \{\phi^h = \tilde{\phi}^h\} = \{\phi^h = \tilde{\phi}^h, \nabla_h \phi^h = \nabla_h \tilde{\phi}^h\} \cup \mathcal{N} \) [11] Theorem 3, Sec. 6], where \( \mathcal{N} \) is a set of measure zero, and \( |\{\phi^h \neq \tilde{\phi}^h\}| \to 0 \) as \( h \to 0 \), we deduce
\[
\lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \text{sym } \nabla_h \phi^h \, dx
\]
\[
= \lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \text{sym } \nabla_h \tilde{\phi}^h \, dx = 0.
\]
Part 2. Again, the equi-integrability property of the sequence \((\Phi^h)\) allows us to modify each \(\Phi^h\) to zero near the boundary, thus, making the following test functions eligible test functions in the Euler–Lagrange equation (32). One easily calculates

\[
\hat{\phi}^h(x) = \left( \Phi^h_{12}(x_1)x_2 + \Phi^h_{13}(x_1)x_3, -\frac{1}{h} \int_0^{x_1} \Phi^h_{12}(s)ds + \Phi^h_{23}(x_1)x_3, -\frac{1}{h} \int_0^{x_1} \Phi^h_{13}(s)ds - \Phi^h_{23}(x_1)x_2 \right),
\]

eligible test functions in the Euler–Lagrange equation (32). One easily calculates

\[
\text{sym } \nabla_h \hat{\phi}^h = \left( \begin{array}{ccc}
(\Phi^h_{12})'(x_1)x_2 + (\Phi^h_{13})'(x_1)x_3 & \frac{1}{2}(\Phi^h_{23})'(x_1)x_3 & -\frac{1}{2}(\Phi^h_{23})'(x_1)x_2 \\
\frac{1}{2}(\Phi^h_{23})'(x_1)x_3 & 0 & 0 \\
-\frac{1}{2}(\Phi^h_{23})'(x_1)x_2 & 0 & 0
\end{array} \right)
\]

Using \(\hat{\phi}^h\) as a test function in (32) together with the symmetry property of the matrix \(DW^h(\cdot, F)F^T\), we obtain

\[
\frac{1}{h^2} \int_\Omega DW^h(x, R^h(I + h^2 G^h)) : \nabla_h \hat{\phi}^h dx
\]

\[
= \frac{1}{h^2} \int_\Omega R^h DW^h(x, I + h^2 G^h)(I + h^2 G^h)^T (R^h)^T : \text{sym } \nabla_h \hat{\phi}^h dx
\]

\[
- \frac{1}{h^2} \int_\Omega R^h DW^h(x, I + h^2 G^h) ((R^h)^T - I + h^2(G^h)^T (R^h)^T) : \nabla_h \hat{\phi}^h dx
\]

\[
= \int_\Omega R^h E^h(I + h^2 G^h)^T (R^h)^T : \text{sym } \hat{\mu}(\Phi^h)' p_{x'} dx
\]

\[
- \int_\Omega R^h E^h \left( \frac{1}{h} ((R^h)^T - I) + h(G^h)^T (R^h)^T \right) : h \nabla_h \hat{\phi}^h dx.
\]

Therefore, the Euler–Lagrange equation becomes

\[
\int_\Omega R^h E^h(I + h^2 G^h)^T (R^h)^T : \text{sym } \hat{\mu}(\Phi^h)' p_{x'} dx
\]

\[
= \int_\Omega R^h E^h \left( \frac{1}{h} ((R^h)^T - I) + h(G^h)^T (R^h)^T \right) : h \nabla_h \hat{\phi}^h dx + \int_\Omega (f_a \hat{\phi}_a^h + f_b \hat{\phi}_b^h) dx.
\]

Since \((h \hat{\phi}_a^h, h \hat{\phi}_b^h) \to 0\) strongly in the \(L^2\)-norm, the force term vanishes at the limit. According to (37), \(\| (\Phi^h)' \|_{L^2(0, L)}\) is uniformly bounded implying the strong convergence \(h \nabla_h \hat{\phi}^h \to 0\) in the \(L^2\)-norm, therefore,

\[
\lim_{h \downarrow 0} \frac{1}{h} \int_\Omega R^h E^h ((R^h)^T - I) : h \nabla_h \hat{\phi}^h dx = 0.
\]
In order to infer zero at the limit as \( h \downarrow 0 \) for the remaining term in (40), namely

\[
h \int_\Omega R^h E^h (G^h)^T (R^h)^T : h \nabla_h \tilde{\phi}^h \, dx,
\]

we need to replace the sequence \((\Phi^h)\) with the one obtained by means of Lemma A.1. We take the sequence \((s^h)\) as above and obtain a sequence \((\tilde{\Phi}^h)\) satisfying \( \| \tilde{\Phi}^h \|_{W^{1,\infty}(0,L)} \leq C s^h \) for some \( C > 0 \). The last bound together with continuous Sobolev embedding \( H^1((0,L), \mathbb{R}^3) \hookrightarrow L^\infty((0,L), \mathbb{R}^3) \) imply \( \| h \nabla_h \tilde{\phi}^h \|_{L^\infty} \leq C \), where, in view of (39), notation \( \tilde{\phi}^h \) is self-explaining.

From the latter we conclude that the second term on the right hand side of (40) vanishes and infer that

\[
\lim_{h \downarrow 0} \int_\Omega R^h E^h (I + h^2 G^h)^T (R^h)^T : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0.
\]

Obviously,

\[
\lim_{h \downarrow 0} \int_\Omega h^2 R^h E^h (G^h)^T (R^h)^T : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0,
\]

and therefore,

\[
\lim_{h \downarrow 0} \int_\Omega R^h E^h (R^h)^T : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0.
\]

Next, we prove that

\[
\lim_{h \downarrow 0} \int_\Omega E^h : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0.
\]

This follows by writing

\[
\int_\Omega E^h : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = \int_\Omega (R^h + (I - R^h)) E^h (R^h + (I - R^h))^T : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx,
\]

and using the convergence result (42) with the fact that \( R^h \to I \) strongly in the \( L^\infty \)-norm.

Now, recall that

\[
A^h (\text{sym} \, \nu(m) + \text{sym} \, \nabla_h \psi^h) = E^h - \frac{1}{h^2} \mathcal{S}^h (\cdot, h^2 G^h) + o^h,
\]

where \( o^h \to 0 \) strongly in the \( L^2 \)-norm. Using truncation arguments on sets \( S^\alpha_h \) and its complement, as in the first part of the proof, we conclude

\[
\lim_{h \downarrow 0} \int_\Omega \frac{1}{h^2} \mathcal{S}^h (\cdot, h^2 G^h) : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0.
\]

Since \( \lim_{h \downarrow 0} \int_\Omega o^h : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0 \), convergence result (43) implies

\[
\lim_{h \downarrow 0} \int_\Omega A^h (\text{sym} \, \nu(m) + \text{sym} \, \nabla_h \psi^h) : \text{sym} \, \nu((\tilde{\phi}^h)' \, p_{x'}) \, dx = 0.
\]
We finalize the proof with a conclusion analogous to the one from Part 1.

3.2. Identification of the limit Euler–Lagrange equations. Let us now more closely identify terms in the Euler-Lagrange equation (32) and consider the limit when \( h \downarrow 0 \). The same reasoning as in Part 2 of the proof of Theorem 3.1 gives the Euler–Lagrange equation (32) in the form

\[
\int_{\Omega} R^h E^h(I + h^2 G^h)^T (R^h)^T : \text{sym} \nabla h \phi \, dx
\]

(45)

for all test functions \( \phi \in H^1(\Omega, \mathbb{R}^3) \) such that \( \phi(0, x') = 0 \) for all \( x' \in \omega \). The aim is now to identify the limit equation in (45) as \( h \downarrow 0 \). Using the facts that, up to a term converging to zero strongly in the \( L^2 \)-norm,

\[
E^h = A^h(\text{sym} \ i(m) + \text{sym} \nabla h \psi^h) + \frac{1}{h^2} \xi^h(\cdot, h^2 G^h),
\]

\( R^h \to I \) strongly in the \( L^\infty \)-norm, and

\[
\lim_{h \downarrow 0} \int_{\Omega} h^2 R^h E^h(G^h)^T (R^h)^T : \text{sym} \nabla h \phi \, dx = 0,
\]

the limit when \( h \downarrow 0 \) (if it exists) of

\[
\int_{\Omega} R^h E^h(I + h^2 G^h)^T (R^h)^T : \text{sym} \nabla h \phi \, dx
\]

equals to the limit

\[
\lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym} \ i(m) + \text{sym} \nabla h \psi^h) : \text{sym} \nabla h \phi \, dx.
\]

The remainder term \( \frac{1}{h^2} \int_{\Omega} \xi^h(\cdot, h^2 G^h) : \text{sym} \nabla h \phi \, dx \) vanishes in the same way as in the proof of Theorem 3.1 and on the limit as \( h \downarrow 0 \), equation (45) reduces to

\[
\lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym} \ i(m) + \text{sym} \nabla h \psi^h) : \text{sym} \nabla h \phi \, dx
\]

(46)

\[
= \lim_{h \downarrow 0} \left( \int_{\Omega} R^h E^h \left( \frac{1}{h}((R^h)^T - I) + h(G^h)^T (R^h)^T \right) : \nabla h \phi \, dx + h \int_{\Omega} (f_2 \phi_2 + f_3 \phi_3) \, dx \right).
\]

First, consider the test function \( \phi(x) = \phi_{11}(x_1) e_1 \), where \( \phi_{11}(0) = 0 \). Since \( \phi_{11,2} = \phi_{11,3} = 0 \), \( \text{sym} \nabla h \phi = \phi'_{11}(x_1) e_1 \otimes e_1 \), and \( h \nabla h \phi \to 0 \) strongly in the \( L^2 \)-norm, (46) amounts to

\[
\lim_{h \downarrow 0} \int_{\Omega} A^h(\text{sym} \ i(m) + \text{sym} \nabla h \psi^h) : \phi'_{11}(x_1) e_1 \otimes e_1 \, dx = 0.
\]
Next, consider test functions of the form \( \phi_i^h(x) = hx_j \phi_{ij}(x_1) e_i \) for \( i = 1, 2, 3 \), \( j = 2, 3 \) and \( i \leq j \), where \( \phi_{ij}(0) = 0 \). Functions \( \phi_i^h \) obviously satisfy \( \phi_{ij,1}^h, h \phi_{ij,2}^h, h \phi_{ij,3}^h \to 0 \) and \( t(\phi_{ij,2}^h, \phi_{ij,3}^h) \to 0 \) strongly in the \( L^2 \)-norm. Calculating

\[
\text{sym} \nabla_h \phi_{ij}^h = \text{sym}(hx_j \phi'_{ij} | \delta_{2j} \phi_{ij} e_i | \delta_{3j} \phi_{ij} e_i),
\]

we easily conclude from (46) that

\[
(48) \lim_{h \to 0} \int_{\Omega} A^h(\text{sym} i(m) + \text{sym} \nabla_h \psi^h) : \phi_{ij}^h(x_1) e_i \otimes e_j \, dx = 0,
\]

for all \( i = 1, 2, 3 \), \( j = 2, 3 \) with \( i \leq j \). Finally, consider the test function given by

\[
\phi^h(x) = \left( \Phi_{12}(x_1) x_2 + \Phi_{13}(x_1) x_3, \frac{1}{h} \int_0^{x_1} \Phi_{21}(s) \, ds + \Phi_{23}(x_1) x_3, \frac{1}{h} \int_0^{x_1} \Phi_{31}(s) \, ds + \Phi_{32}(x_1) x_2 \right),
\]

where \( \Phi : [0, L] \to \mathbb{R}^{3 \times 3} \) and \( \Phi(0) = 0 \). On the right hand side of (46), using the convergence results: \( R^h \to I \) strongly in the \( L^\infty \)-norm, \( hG^h \to 0 \) strongly in the \( L^2 \)-norm, \( A^h \to A \) strongly in the \( L^\infty \)-norm, as well as the approximation identity (35) for \( E^h \), we are left with

\[
\lim_{h \to 0} \int_{\Omega} A^h(\text{sym} i(m) + \text{sym} \nabla_h \psi^h) A^T : \Phi \, dx
\]

\[
+ \int_0^L \left( f_2(x_1) \int_0^{x_1} \Phi_{21}(s) \, ds + f_3(x_1) \int_0^{x_1} \Phi_{31}(s) \, ds \right) \, dx_1.
\]

Let us now consider the first term of the obtained expression. Due to the real matrix identity \( XY : Z = -X :ZY \), for \( Y \) being skew-symmetric matrix, the first term equals

\[
\lim_{h \to 0} \int_{\Omega} A^h(\text{sym} i(m) + \text{sym} \nabla_h \psi^h) : \Phi A \, dx,
\]

and since the first matrix is symmetric, the latter in fact equals to

\[
(49) \lim_{h \to 0} \int_{\Omega} A^h(\text{sym} i(m) + \text{sym} \nabla_h \psi^h) : \text{sym}(\Phi A) \, dx.
\]

The matrix \( \Phi A \) can be explicitly computed, and its symmetric part is given by

\[
\text{sym}(\Phi A) = \begin{pmatrix}
\frac{1}{2} (\Phi_{12} v'_2 + \Phi_{13} v'_3) & \frac{1}{2} (\Phi_{12} v'_2 + \Phi_{13} w) & -\frac{1}{2} (\Phi_{23} v'_2 + \Phi_{12} w) \\
\frac{1}{2} (\Phi_{23} v'_3 + \Phi_{13} w) & \Phi_{12} v'_2 + \Phi_{13} v'_3 & \frac{1}{2} (\Phi_{13} v'_2 + \Phi_{12} v'_3) \\
-\frac{1}{2} (\Phi_{23} v'_2 + \Phi_{12} w) & \frac{1}{2} (\Phi_{13} v'_2 + \Phi_{12} v'_3) & \Phi_{13} v'_3 + \Phi_{23} w
\end{pmatrix}.
\]

Defining the sequence of test functions \( (\varphi_A^h) \) by

\[
\varphi_A^h(x) = hx_2 \begin{pmatrix}
\Phi_{23} v'_3 + \Phi_{13} w \\
\Phi_{12} v'_2 + \Phi_{13} v'_3 \\
\Phi_{13} v'_2 + \Phi_{12} v'_3
\end{pmatrix} + hx_3 \begin{pmatrix}
-\Phi_{23} v'_2 - \Phi_{12} w \\
\Phi_{13} v'_2 + \Phi_{12} v'_3 \\
\Phi_{13} v'_3 + \Phi_{23} w
\end{pmatrix},
\]
it is straightforward to check that

\begin{equation}
\text{sym}(\Phi A) = \text{sym} \nabla_h \Phi^h + (\Phi_{12} v'_2 + \Phi_{13} v'_3) e_1 \otimes e_1 + o^h,
\end{equation}

where \( o^h \) converges to zero strongly in the \( L^2 \)-norm as \( h \downarrow 0 \). Observe that the sequence of test functions \( (\varphi^h_A) \) satisfies \( (\varphi^h_{A,1}, h \varphi^h_{A,2}, h \varphi^h_{A,3}) \rightarrow 0 \) and \( t(\varphi^h_{A,1}, \varphi^h_{A,3}) \rightarrow 0 \) strongly in the \( L^2 \)-norm. Utilizing (50) in expression (49), we confer that due to the orthogonality property (36), convergence result (47) and strongly to zero convergence of \( o^h \), these terms vanish in the limit as \( h \downarrow 0 \). Since,

\begin{equation}
\text{sym} \nabla_h \phi^h = 
\begin{pmatrix}
\Phi'_1(x_1)x_2 + \Phi'_3(x_1)x_3 & \frac{1}{2} \Phi'_2(x_1)x_3 & -\frac{1}{2} \Phi'_2(x_1)x_2 \
\frac{1}{2} \Phi'_2(x_1)x_3 & 0 & 0 \\
-\frac{1}{2} \Phi'_2(x_1)x_2 & 0 & 0
\end{pmatrix} &= \text{sym} t((\Phi')p_{x'}),
\end{equation}

the left hand side in (46) can be written as

\begin{equation}
\lim_{h \downarrow 0} \int_{\Omega} A^h (\text{sym} t(m) + \text{sym} \nabla_h \psi^h) : \text{sym} (\sum_{i \leq j} \phi'_{ij} e_i \otimes e_j + t(\Phi'p_{x'})) \, dx.
\end{equation}

Combining (47), (48) and (52), the resolved limiting Euler–Lagrange equation (46) reads

\begin{equation}
\lim_{h \downarrow 0} \int_{\Omega} A^h (\text{sym} t(m) + \text{sym} \nabla_h \psi^h) : \text{sym} (\sum_{i \leq j} \phi'_{ij} e_i \otimes e_j + t(\Phi'p_{x'})) \, dx
\end{equation}

\begin{equation}
= -\int_0^L (f_2 \Phi_{12} + f_3 \Phi_{13}) \, dx_1,
\end{equation}

where \( \Phi_{1j}(x_1) = \int_0^{x_1} \varphi_{1j} (s) \, ds \) for \( j = 2, 3 \). Now, observe that the obtained equation (after removing terms \( \phi'_{22} e_2 \otimes e_2, \phi'_{23} e_2 \otimes e_3 \) and \( \phi'_{33} e_3 \otimes e_3 \) in the first sum) can be interpreted as

\begin{equation}
\frac{\delta K(\psi)}{\delta m} \left[ \sum_{j=1,2,3} \phi'_{1j} e_j + \Phi'p_{x'} \right] = -\int_0^L (f_2 \Phi_{12} + f_3 \Phi_{13}) \, dx_1,
\end{equation}

which is exactly the variation of the functional \( K^0 \). Since \( (\text{sym} \nabla_h \psi^h) \) is bounded in the \( L^2 \)-norm, according to [19 Lemma 2.17], there exists a subsequence (still denoted by \( (h) \)) and sequence \( (\tilde{\psi}^h) \) such that \( (\text{sym} \nabla_h \tilde{\psi}^h)^2 \) is equi-integrable and \( \| \text{sym} \nabla_h \psi^h - \text{sym} \nabla_h \tilde{\psi}^h \|_{L^2(\Omega^h)} \rightarrow 0 \), where \( \Omega^h \subset \Omega \) such that \( |\Omega \setminus \Omega^h| \rightarrow 0 \). From (53) we see that the same limit equation will be obtained if we replace the relaxation sequence \( (\psi^h) \) by \( (\tilde{\psi}^h) \).

Stationarity of the point \( (u, v_2, v_3, w) \) for the functional \( K^0 \) is (up to the linear force term) equivalent to the stationarity of \( m \) (defined by (21)) for the functional \( K(\psi) \). Let \( (\psi^h_m) \) be the relaxation sequence for \( m \) from Lemma 2.3 Using the coercivity of \( Q^h \) and the orthogonality...
properties (28) and (36) of both sequences \((\psi_m^h)\) and \((\tilde{\psi}^h)\), respectively, we find that
\[
\alpha \|\text{sym } \nabla_h (\psi_m^h - \tilde{\psi}^h)\|_{L^2}^2 \leq \int_{\Omega} Q_h(x, \text{sym } \nabla_h (\psi_m^h - \tilde{\psi}^h)) \, dx
\]
\[
= \frac{1}{2} \int_{\Omega} A_h(x, \text{sym } \iota(m) + \text{sym } \nabla_h \psi_m^h) : \text{sym } \nabla_h (\psi_m^h - \tilde{\psi}^h) \, dx
\]
\[
- \frac{1}{2} \int_{\Omega} A_h(x, \text{sym } \iota(m) + \text{sym } \nabla_h \psi^h) : \text{sym } \nabla_h (\psi_m^h - \tilde{\psi}^h) \, dx \rightarrow 0
\]
as \(h \downarrow 0\). Therefore, we can also replace the sequence \((\tilde{\psi}^h)\) by \((\psi_m^h)\) and according to (31), \(m\) is indeed the stationary point of the limit functional \(K_h\). This finishes the proof of Theorem 1.1.

In the subsequent part of the section we identify the limit Euler–Lagrange equations. Recalling the approximation identity (35), the weak convergence \(E_h \rightharpoonup E\) in \(L^2(\Omega, \mathbb{R}^{3 \times 3})\), and utilizing convergence properties for the remainder terms, we can pass to the limit in equation (53) and obtain
\[
\int_{\Omega} E : \text{sym } \left( \sum_{i \leq j} \phi_{ij}^e e_i \otimes e_j \right) 
= -\int_0^L (f_2 \tilde{\Phi}_{12} + f_3 \tilde{\Phi}_{13}) \, dx_1.
\]
In view of identity (51), the latter equals
\[
\int_{\Omega} \left( \sum_{i \leq j} E_{ij} \phi_{ij}^e + x_2 E_{11} \Phi_{12}'(x_1) + x_3 E_{11} \Phi_{13}'(x_1) + x_3 E_{12} \Phi_{23}'(x_1) - x_2 E_{13} \Phi_{23}'(x_1) \right) \, dx
\]
\[
= -\int_0^L (f_2 \tilde{\Phi}_{12} + f_3 \tilde{\Phi}_{13}) \, dx_1.
\]
Using the moment notation (10)–(11) and the fact that \(\tilde{\Phi}'_{ij} = \Phi_{ij}\) for \(j = 2, 3\), (54) becomes
\[
\int_0^L \left( \sum_{i \leq j} \tilde{E}_{ij} \phi_{ij}^e + \tilde{E}_{11} \tilde{\Phi}_{12}'' + \tilde{E}_{11} \tilde{\Phi}_{13}'' + \tilde{E}_{12} \Phi_{23}' - \tilde{E}_{13} \Phi_{23}' \right) \, dx_1
\]
\[
= -\int_0^L (f_2 \tilde{\Phi}_{12} + f_3 \tilde{\Phi}_{13}) \, dx_1.
\]

Now by the arbitrariness of test functions, we easily derive the corresponding strong formulation for the moments. The zeroth-order moments satisfy
\[
\tilde{E} = 0 \quad \text{in } (0, L).
\]
First-order moments \(\tilde{E}_{11}\) and \(\tilde{E}_{11}\) satisfy second-order boundary-value problems:
\[
\tilde{E}_{11}'' + f_2 = 0 \quad \text{in } (0, L),
\]
\[
\tilde{E}_{11}(L) = \tilde{E}_{11}'(L) = 0.
\]
and
\begin{align}
\widehat{E}_{11}^\prime + f_3 &= 0 \quad \text{in } (0, L), \\
\widehat{E}_{11}(L) &= \widehat{E}_{11}^\prime(L) = 0,
\end{align}
respectively. Finally, first-order moments $\widehat{E}_{12}$ and $\widehat{E}_{13}$ satisfy the first-order problem
\begin{align}
\widehat{E}_{12}^\prime - \widehat{E}_{13}^\prime &= 0 \quad \text{in } (0, L), \\
\widehat{E}_{12}(L) &= \widehat{E}_{13}(L).
\end{align}

It remains to derive constitutive equations, which connect the moments of the limit stress with limit displacements and twist functions. For $\varrho \in L^2(0, L)$ and $\Psi \in L^2((0, L), \mathbb{R}^{3\times3})$, recall the functional
\begin{equation}
K(h)(m(\varrho, \Psi)) = \int_0^L Q_0^0(x_1, \varrho(x_1), \text{axl } \Psi(x_1))dx_1,
\end{equation}
where $m(\varrho, \Psi)(x) = \varrho(x_1)e_1 + \Psi(x_1)p_x$, and the functional
\begin{align}
K(h)(\Psi) &= \int_0^L Q_1^0(x_1, \text{axl } \Psi(x_1))dx_1 = \int_0^L Q_0^0(x_1, \varrho_0(x_1), \text{axl } \Psi(x_1))dx_1 \\
&= K(h)(m_0(\varrho_0, \Psi)),
\end{align}
where $\varrho_0 : (0, L) \times \mathbb{R}^3 \to \mathbb{R}$ is the optimal for a given axl $\Psi$. By Lemma 2.3 (identity (27)), there exist sequences $(\psi^h_m) \subset H^1(\Omega, \mathbb{R}^3)$ and $(\psi_0^h) \subset H^1(\Omega, \mathbb{R}^3)$ such that:
\begin{align}
K(h)(m(\varrho, \Psi)) &= \lim_{h \to 0} \int_\Omega Q^h(x, \text{sym } m + \text{sym } \nabla_h \psi^h_m)dx, \\
K(h)(\Psi) &= \lim_{h \to 0} \int_\Omega Q^h(x, \text{sym } m_0 + \text{sym } \nabla_h \psi^h_0)dx.
\end{align}
Using the orthogonality property (28) and tricks as in Section 2.6, we calculate:
\begin{align}
\delta K(h)(m(\varrho, \Psi)) \frac{\delta}{\delta \varrho} [\phi] &= \lim_{h \to 0} \int_\Omega A^h(x)(\text{sym } m + \text{sym } \nabla_h \psi^h_m) : \nabla(\phi e_1)dx, \\
\delta K(h)(\Psi) \frac{\delta}{\delta \Psi} [\Phi] &= \lim_{h \to 0} \int_\Omega A^h(x)(\text{sym } m_0 + \text{sym } \nabla_h \psi^h_0) : \nabla(\Phi p_x)dx,
\end{align}
for all $\phi \in C^\infty_0(0, L)$ and $\Phi \in C^\infty_0((0, L), \mathbb{R}^{3\times3})$. On the other hand, from the representation of function $Q_1^0$ as a pointwise quadratic form, we have
\begin{equation}
\delta K(h)(\Psi) \frac{\delta}{\delta \Psi} [\Phi] = \int_0^L A_1^0(x_1) \text{axl } \Psi(x_1) \cdot \text{axl } \Phi(x_1)dx_1.
\end{equation}
Now, if we consider $m(x) = (u' + \frac{1}{2}((v_2')^2 + (v_3')^2))e_1 + A'p_{x'}$, it follows from formulae (60) and (47) that
\[
\frac{\delta K_{(h)}(m(a, A'))}{\delta \phi}[\phi] = 0
\]
for all $\phi \in C_0^\infty(0, L)$, where $a(x_1) = u' + \frac{1}{2}((v_2')^2 + (v_3')^2)$. In particular, this implies the optimality of function $a$ for matrix function $A'$ in the sense that $Q_1^h(\cdot, axl A') = Q_1^h(\cdot, a, axl A')$. Equating expressions in (61) and (62) for $\Psi = A'$ and $\varrho_0 = a$, we obtain the identity
\[
\int_0^L A_1^0(x_1) axl A'(x_1) \cdot axl \Phi(x_1) dx_1 = \int_\Omega E : i((axl A') dx_1,
\]
for all $\Phi \in C_0^\infty((0, L), \mathbb{R}^{3 \times 3}_{skw})$. From the latter we recognize the following system
\[
-(A_1^0 axl A')_3 = \widetilde{E}_{11}, \\
(A_1^0 axl A')_2 = \widetilde{E}_{11}, \\
-(A_1^0 axl A')_1 = \widetilde{E}_{12} - \widetilde{E}_{13},
\]
which is a linear second-order system for the limit displacements $v_2, v_3$ and the limit twist function $w$, and which needs to be accompanied by the following boundary conditions $v_i(0) = v_i'(0) = 0$ for $i = 2, 3$, and $w(0) = 0$. The obtained boundary-value problem represents the homogenized Euler–Lagrange equations for the von Kármán rod model. Finally, the scaled displacement $u$ can be deduced from the optimality property of the function $a$ for the matrix function $A'$ and the initial condition $u(0) = 0$.

APPENDIX A.

Lemma A.1. Let $p > 1$, $\Omega \subset \mathbb{R}^d$ open, bounded set and $(u_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)$ a bounded sequence such that $(|\nabla u_k|^p)$ is equi-integrable. Let $(s_k)_k$ be an increasing sequence of positive reals such that $s_k \to +\infty$ for $k \to +\infty$. Then there exists a subsequence still denoted by $(u_k)$ and a sequence $(z_k) \subset W^{1,\infty}(\Omega, \mathbb{R}^m)$ satisfying: $|z_k - u_k| \to 0$ as $k \to +\infty$, $(|\nabla z_k|^p)$ is equi-integrable and $\|z_k\|_{W^{1,\infty}} \leq C s_k$ for some $C > 0$ depending only on dimension $d$.

Proof. The proof is implicitly contained in the proof of Lemma 1.2 (decomposition lemma) from [14], but we include it here for reasons of completeness. As in [14], the proof is divided into two steps. In the first we assume that $\Omega$ is a extension domain, while in the second we remove this restriction generalizing the statement for an arbitrary open set.

Step 1. Let $\Omega \subset \mathbb{R}^d$ be an extension domain, i.e. an open, bounded set for which there exists an extension operator $T_\Omega : W^{1,p}(\Omega, \mathbb{R}^m) \to W^{1,p}(\mathbb{R}^d, \mathbb{R}^m)$ satisfying:
\[
T_\Omega u = u \quad \text{on } \Omega, \quad \|T_\Omega u\|_{W^{1,p}} \leq C \|u\|_{W^{1,p}}.
\]
In the following we identify the sequence \((u_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)\) with its extension sequence \((T_\Omega u_k) \subset W^{1,p}(\mathbb{R}^d, \mathbb{R}^m)\). Let us introduce the Hardy-Littlewood maximal function

\[
M(u)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|dy,
\]

defined for any Borel measurable function \(u : \mathbb{R}^d \to \mathbb{R}^m\). It is known that for \(p > 1\) and \(u \in W^{1,p}(\mathbb{R}^d, \mathbb{R}^m)\),

\[
\|M(u)\|_{L^p} + \|M(\nabla u)\|_{L^p} \leq C\|u\|_{W^{1,p}}.
\]

According to [14, Lemma 4.1] (cf. [11]), for every \(k \in \mathbb{N}\), there exists \(z_k \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^m)\) such that \(u_k = z_k\) on the set \(S_k := \{M(\nabla u_k)(x) < s_k\}\) and \(\|z_k\|_{W^{1,\infty}} \leq C s_k\), where \(C > 0\) depends only on \(d\). Using the Chebyshev inequality and uniform bound (64), we obtain the estimate on the Lebesgue measure of the complement of set \(S_k\),

\[
|S_k^c| \leq \frac{1}{s_k} \int_{\Omega} M(\nabla u_k)dx \leq \frac{C}{s_k}, \quad \text{for all } k \in \mathbb{N}.
\]

Hence, \(|S_k^c| = |u_k \neq z_k| \to 0\) as \(k \to \infty\). Due to the fact that \(\{u_k = z_k\} = \{u_k = z_k, \nabla u_k = \nabla z_k\}\), up to a set of the Lebesgue measure zero, for a.e. \(x \in S_k\), we have

\[
|\nabla z_k(x)| = |\nabla u_k(x)| \leq M(\nabla u_k)(x).
\]

On the other hand, for a.e. \(x \in S_k^c\), it holds

\[
|\nabla z_k(x)| \leq C s_k \leq C M(\nabla u_k)(x).
\]

Thus, we conclude that \(|\nabla z_k(x)|^p \leq C|M(\nabla u_k)(x)|^p\) for a.e. \(x \in \Omega\), which together with the uniform bound (64) implies the equi-integrability property of the sequence \(|\nabla z_k|^p\).

Step 2. Let \(\Omega\) be an arbitrary open, bounded set. For a given bounded sequence \((u_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)\), there exists a subsequence such that

\[
u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m), \quad u_k \to u \text{ in } L^p_{\text{loc}}(\Omega, \mathbb{R}^m).
\]

Let \((\Omega_l)\) be an increasing sequence of compactly contained subdomains of \(\Omega\) satisfying \(|\Omega \setminus \Omega_l| \to 0\) as \(l \to \infty\), and let \((\zeta_l) \subset C_0^\infty(\Omega, [0,1])\) be a sequence of cut-off functions such that \(\zeta_l(x) = 1\) for \(x \in \Omega_l\). Define \(\tilde{u}_k := u_k - u\), and observe that

\[
\limsup_{l \to \infty} \limsup_{k \to \infty} \|\zeta_l \tilde{u}_k\|_{L^p} = 0
\]

and

\[
\limsup_{l \to \infty} \limsup_{k \to \infty} \|\nabla(\zeta_l \tilde{u}_k)\|_{L^p} = \limsup_{l \to \infty} \limsup_{k \to \infty} \|\nabla \zeta_l \otimes \tilde{u}_k + \zeta_l \nabla \tilde{u}_k\|_{L^p}
\]

\[
\leq \limsup_{k \to \infty} \|\nabla \tilde{u}_k\|_{L^p} < \infty.
\]

Then, a standard diagonalization procedure applies (cf. [2, Lemma 1.15]) and provides a bounded sequence \((\zeta_l(x) \tilde{u}_k) \subset W_0^{1,p}(\Omega, \mathbb{R}^m)\), which can be extended by zero to \(\mathbb{R}^d\). Since,
\((|\nabla (\zeta (k) \tilde{u}_k)|^p)\) is equi-integrable, applying the arguments of Step 1, there exists a sequence \(\tilde{z}_k \subset W^{1,p}(\Omega, \mathbb{R}^m)\) satisfying: \(|\tilde{z}_k \neq \zeta (k) \tilde{u}_k| \to 0\) as \(k \to +\infty\), \((|\nabla \tilde{z}_k|^p)\) is equi-integrable and \(\|\tilde{z}_k\|_{W^{1,\infty}} \leq C s_k\) for some \(C > 0\). Since, \(|\tilde{z}_k + u \neq u_k| \leq |\tilde{z}_k \neq \zeta (k) \tilde{u}_k| + |\Omega \setminus \Omega (k)| \to 0\), \((|\nabla (\tilde{z}_k + u)|^p)\) is equi-integrable, and \(\|\tilde{z}_k + u\|_{W^{1,\infty}} \leq C s_k\) for some \(C > 0\), we identify \(z_k = \tilde{z}_k + u\) as the sought sequence. \(\square\)

**Remark A.1.** If we assume in the previous lemma that \(\Omega\) is a Lipschitz domain, as it is the case in our model of the rod, where \(\Omega = (0, L) \times \omega\) and \(\omega\) is Lipschitz, then \(\Omega\) is also an extension domain and according to the arguments in Step 1, we can replace the whole sequence \((u_k)\) by its Lipschitz counterpart.

The following corollary provides the analogous statement to the previous lemma, but with the gradients replaced by the scaled gradients. A general idea for proving such results can be found in \([6]\) (cf. also \([19, proof of Lemma 2.17]\)), therefore, we omit the proof here.

**Corollary A.2.** Let \(p > 1\), \(\Omega \subset \mathbb{R}^d\) open, bounded set, \((h_k)\) monotonically decreasing to zero sequence of positive numbers, and \((u_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)\) a bounded sequence such that \((\nabla h_k u_k)\) is bounded in \(L^p(\Omega, \mathbb{R}^m)\) and \((|\nabla h_k u_k|^p)\) is equi-integrable. Let \((s_k)_k\) be an increasing sequence of positive reals such that \(s_k \to +\infty\) for \(k \to +\infty\). Then, there exists a subsequence still denoted by \((u_k)\) and a sequence \((z_k) \subset W^{1,\infty}(\Omega, \mathbb{R}^m)\) satisfying: \(|z_k \neq u_k| \to 0\) as \(k \to +\infty\), \((|\nabla h_k z_k|^p)\) is equi-integrable, \(|z_k|_{W^{1,\infty}} \leq C s_k\) and \(\|\nabla h_k z_k\|_{L^\infty} \leq C s_k\) for some \(C > 0\) depending only on dimension \(d\).

**References**


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