ON THE GENERAL HOMOGENIZATION OF VON KÁRMÁN PLATE EQUATIONS FROM 3D NONLINEAR ELASTICITY

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Abstract. Starting from 3D elasticity equations we derive the model of the homogenized von Kármán plate by means of Γ-convergence. This generalizes the recent results, where the material oscillations were assumed to be periodic.

Keywords: elasticity, dimension reduction, homogenization, von Kármán plate model.

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This paper is about the derivation of homogenized von Kármán plate equations, starting from 3D elasticity by means of Γ-convergence. We do not presuppose any kind of periodicity, but work in a general framework. There is a vast literature on deriving plate equations from 3D elasticity. For the approach using formal asymptotic expansion, see [Cia97] and the references therein. The first work on deriving the plate models by means of Γ-convergence was [LDR95], where the authors derived the membrane plate model. It was well known that the obtained models depend on the assumption on the relation of the external loads (i.e., the energy) with respect to the thickness of the body $h$. Higher order models (such as bending and von Kármán plate models) were also derived by means of Γ-convergence (see [FJM02, FJM06]). The key mathematical ingredient in these cases was the theorem on geometric rigidity.

In [BFF00] (see also [BB06]) the influence of different inhomogeneities, in combination with dimensional reduction, on the limit model was analyzed. These models were obtained in the membrane regime. Recently, the techniques from [FJM02, FJM06] were combined together with two-scale convergence to obtain the models of homogenized rod in the bending regime (see [Nie12], see also [Nie10]), homogenized von Kármán plate (see [Vel13, NV13]), homogenized von Kármán shell (see [HV14]) and homogenized bending plate (see [HN14, Vel15]). These models were derived under the assumption of the periodic oscillations, where it was assumed that the material oscillates on the scale $\varepsilon(h)$, while the thickness of the body is $h$. The obtained models depend on the parameter $\gamma = \lim_{h \to 0} \frac{h}{\varepsilon(h)}$. In the case of bending rod and von Kármán plate the situation $\gamma = 0$ corresponds to the case when dimensional reduction dominates and the obtained model is the model of homogenized bending rod (von Kármán plate) and can be derived as the limit case when $\gamma \to 0$. Analogously, the situation when $\gamma = \infty$ corresponds to the case when homogenization dominates and can again be derived as the limit when $\gamma \to \infty$; this is the model of bending rod (von Kármán plate) obtained starting from homogenized energy. In the case of von Kármán shell and bending plate the situation $\gamma = 0$ was more subtle and led to the models depend on the further assumption on the relation between $\varepsilon(h)$ and $h$. We obtained different models for the case $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ and $h \sim \varepsilon(h)^2$. Moreover, the model obtained by homogenizing bending plate (2D) does not have the energy density of the same form as models obtained for $\gamma > 0$ or $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ (see [NO15]). In this case the homogenization is done under the constraint of being an isometry.

In this paper we analyze the case of the simultaneous homogenization and dimensional reduction in von Kármán regime in the general framework, without any assumption on the periodicity. Moreover, we do not suppose that the oscillations of the material are only in the in-plane directions (as opposed to periodic oscillations in [NV13]), but can happen in any direction (even cross-sectional). We obtain kind of stability result for the equations, i.e., in the limit, we always obtain the equations of von Kármán type. Simultaneous homogenization and dimensional reduction, without any assumption of periodicity, was also considered in a non-variational framework (see [CM04] for monotone nonlinear elliptic systems and [CM06] for linear elasticity system). In these papers compactness arguments were used and the notion of $H$-convergence (introduced by Murat and Tartar, see [MT97]) was adapted to the dimensional reduction.

This paper is the first treatment of simultaneous homogenization and dimensional reduction, without periodicity assumption, by variational techniques, in the context of higher order models in elasticity, at least to the author’s knowledge (membrane case is already analyzed in [BFF00]). The case of bending rod model has also recently been analyzed by...
using the approach developed here (see [MY15]). That is the generalization of the periodic case analyzed in [Neu12]. Here we restrict ourselves to the von Kármán plate model, where the linearization is already dominated, although the system itself is nonlinear. The case of bending plate and von Kármán shell is not possible to treat with this approach (and in the author’s opinion these equations are not stable in the sense explained above), due to more complex phenomenology in the periodic case, as explained above.

We prove the validity of the following asymptotic formula for the energy density

\[
Q(x'_0, M_1, M_2) = \lim_{h \to 0} \frac{1}{|B(x'_0, r)|} K \left( M_1 + x_3 M_2, B(x'_0, r) \right),
\]

\[
\forall M_1, M_2 \in \mathbb{R}^{2 \times 2}_{\text{sym}} \text{ and a.e. } x'_0 \in \omega,
\]

where

\[
K \left( M_1 + x_3 M_2, B(x'_0, r) \right) = \inf \left\{ \lim_{h \to 0} \inf_{\psi \in B(x'_0, r) \times I} \int_{B(x'_0, r) \times I} Q^h \left( x, \iota(M_1 + x_3 M_2) + \nabla h \psi^h \right) \, dx : \right.
\]

\[
\left. \psi^h_1, \psi^h_2, h \psi^h_3 \to 0 \text{ strongly in } L^2 \left( B(x'_0, r) \times I, \mathbb{R}^3 \right) \right\},
\]

\[
B(x'_0, r) \text{ is a ball of radius } r \text{ and center } x'_0 \text{ in } \mathbb{R}^2, \omega \subset \mathbb{R}^2 \text{ is a Lipschitz domain, representing the plate, } I = (-\frac{1}{2}, \frac{1}{2}), x = (x', x_3), \iota \text{ is the natural injection from } \mathbb{R}^{2 \times 2} \text{ to } \mathbb{R}^{3 \times 3} \text{ (see below), } (Q^h)_{h>0} \text{ are quadratic functionals of } h \text{ problem (see Section 2) and } \nabla h \psi^h = (\partial_1 \psi^h, \partial_2 \psi^h, \frac{1}{h} \partial_3 \psi^h). \text{ This formula is actually valid on a subsequence and unifies all three regimes obtained in } [NV13]. \text{ For the form of the limit energy in the case analyzed here, see also the expression [11] below.}
\]

We emphasize the fact that in the construction of recovery sequence for the limit equations we do not use any kind of additional regularity of minimizers (or their higher integrability, see [SW94]), that is usually done, but only equi-integrability of the minimizing sequence. Besides equi-integrability of the minimizing sequence (i.e., the possibility to replace the gradients and scaled gradients by equi-integrable ones), there are two more key points of the proofs. One is the characterization of the displacements that have bounded symmetric gradients on thin domains (we adapt the result proved in [Gri05, Theorem 2.3]) and the other is the characterization of the displacements that have the energy of order $h^4$ (see [NV13, Proposition 3.1]). The proof of this proposition relied on the theorem on geometric rigidity and similar observations in [FJM06]. The new essential part was to correct the vertical displacements in order to obtain a sequence which is bounded in $H^2$. This was not done in [FJM06], since the authors did not need more information on the corrector to obtain the lower bound.

The main results of this paper is Theorem 2.6. It is the analogue of standard Γ-compactness result. We show that, under von Kármán scaling assumption, we always obtain, on a subsequence, von Kármán type energy, where the energy density is given by the above asymptotic formula.

The novelty of this paper consists in recognizing the above asymptotic formula, but also in the adaptation of the general Γ-convergence techniques to the case of higher order thin models in elasticity. Although the basic approach is not difficult (see the end of Section 2 where the strategy of the proofs is explained), it has its own peculiarities.

This paper is organized as follows: in Section 2 we give general framework and main result, we also explain the strategy of the proofs, in Section 3 we give the proofs of the main statements given in Section 2, in Section 4 we apply the obtain results to periodic oscillations in the direction of thickness coupled with in-plane oscillations and in the Appendix we prove some auxiliary claims.
1.1. Notation. If \( x \in \mathbb{R}^3 \) by \( x' \) we denote \( x' = (x_1, x_2) \). By \( \nabla' \) we denote the operator \( \nabla' u = (\partial_1 u, \partial_2 u) \) and by \( \nabla_h \) the operator \( \nabla_h u = (\partial_1 u, \partial_2 u, \frac{1}{h} \partial_3 u) \). By \( B(x, r) \) we denote the ball of radius \( r \) with the center \( x \). If \( A \subset \mathbb{R}^n \), by \( |A| \) we denote the Lebesgue measure of \( A \) and by \( \chi_A \) we denote the characteristic function of the set \( A \). If \( A \) and \( B \) are subsets of \( \mathbb{R}^n \), by \( A \ll B \) we mean that the closure \( \bar{A} \) is contained in the interior \( \text{int}(B) \) of \( B \). \( t \) denotes the natural injection of \( \mathbb{R}^{2 \times 2} \) into \( \mathbb{R}^{3 \times 3} \). Denoting the standard basis of \( \mathbb{R}^3 \) by \((e_1, e_2, e_3)\) it is given by

\[
\iota(A) := \sum_{\alpha, \beta=1}^{2} A_{\alpha\beta}(e_\alpha \otimes e_\beta).
\]

For \( a, b \in \mathbb{R}^3 \) by \( a \wedge b \) we denote the wedge product of the vectors \( a \) and \( b \). For \( a, b \in \mathbb{R}^n \) with \( a \otimes b \) we denote the standard tensor product (an element of \( \mathbb{R}^{n \times n} \)). For \( A \in \mathbb{R}^{n \times n} \) by \( A^t \) we denote its transpose, while sym \( A \) denotes its symmetric part \( \text{sym} A = \frac{1}{2}(A + A^t) \). By \( \mathbb{R}_{\text{sym}}^{n \times n} \) we denote the space of symmetric matrices of order \( n \) while by \( \mathbb{R}_{\text{skw}}^{n \times n} \) we denote the space of antisymmetric matrices of order \( n \). By \( \text{Id} \) we denote the identity matrix.

As usual, by \( W^{k,p}(S, \mathbb{R}^n) \) we denote the Sobolev space with the domain that is an open subset \( S \subset \mathbb{R}^m \) and that take values in \( \mathbb{R}^n \) and that have first \( k \) derivatives which are \( p \)-integrable. When \( p = 2 \) we denote this space by \( H^k(S, \mathbb{R}^n) \). When we want to emphasize the dependence of a sequence on variable \( h \) we sometimes superscript \( h \), sometimes superscript \( h_n \). In the second case we want to emphasize the importance of depending on some predefined sequence (see, e.g., Theorem 2.12, Theorem 2.13), while in the first case such dependence is not so important and claims are satisfied for any sequence or even every \( h \) (see, e.g., Theorem 3.1, Lemma 3.2).

2. General framework and main results

The three-dimensional model. Throughout the paper \( \Omega^h := \omega \times (hI) \) denotes the reference configuration of a thin plate with mid-surface \( \omega \subset \mathbb{R}^2 \) and (rescaled) cross-section \( I := (-\frac{1}{2}, \frac{1}{2}) \). We suppose that \( \omega \) is Lipschitz domain, i.e., open, bounded and connected set with Lipschitz boundary. For simplicity we assume that \( \omega \) is centered, that is

\[
\int_{\omega} \left( \begin{array}{c} x_1 \\ x_2 
\end{array} \right) \, dx_1 \, dx_2 = 0.
\]

The canonical domain \( \Omega \) is simply denoted by \( \Omega \). In the von Kármán regime we define the energy functionals (these functionals are obtained after rescaling on the canonical domain and after neglecting the term with forces)

\[
I^h(y) := \frac{1}{h^4} \int_{\Omega} W^h(x, \nabla_y(x)) \, dx,
\]

imposing their finiteness. Here \( y \in H^1(\Omega, \mathbb{R}^3) \) is a deformation and \( W^h \) is the stored energy density with the properties given below. The von Kármán regime is characterized by the fact that we assume that the energy has the order \( h^4 \).

Definition 2.1 (nonlinear material law). Let \( 0 < \eta_1 \leq \eta_2 \) and \( \rho > 0 \). The class \( \mathcal{W}(\eta_1, \eta_2, \rho) \) consists of all measurable functions \( W : \mathbb{R}^{3 \times 3} \to [0, +\infty] \) that satisfy the
following properties:

(W1) \( W \) is frame indifferent, i.e.
\[
W(RF) = W(F) \quad \text{for all } F \in \mathbb{R}^{3 \times 3}, R \in \text{SO}(3);
\]

(W2) \( W \) is non-degenerate, i.e.
\[
\begin{align*}
W(F) &\geq \eta_1 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3}; \\
W(F) &\leq \eta_2 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ with } \text{dist}^2(F, \text{SO}(3)) \leq \rho;
\end{align*}
\]

(W3) \( W \) is minimal at \( \text{Id} \), i.e.
\[
W(\text{Id}) = 0;
\]

(W4) \( W \) admits a quadratic expansion at \( \text{Id} \), i.e.
\[
W(Id + G) = Q(G) + o(|G|^2) \quad \text{for all } G \in \mathbb{R}^{3 \times 3},
\]
where \( Q : \mathbb{R}^{3 \times 3} \to \mathbb{R} \) is a quadratic form.

In the following definition we state our assumptions on the family \((W^h)_{h>0}\).

**Definition 2.2** (admissible composite material). Let \( 0 < \eta_1 \leq \eta_2 \) and \( \rho > 0 \). We say that a family \((W^h)_{h>0}\)
\[
W^h : \Omega \times \mathbb{R}^{3 \times 3} \to [0, +\infty]
\]
describes an admissible composite material of class \(W(\eta_1, \eta_2, \rho)\) if

(i) for each \( h > 0 \), \( W^h \) is almost surely equal to a Borel function on \( \Omega \times \mathbb{R}^{3 \times 3} \);
(ii) \( W^h(x, \cdot) \in W(\eta_1, \eta_2, \rho) \) for every \( h > 0 \) and almost every \( x \in \Omega \);
(iii) there exists a monotone function \( r : \mathbb{R}^+ \to [0, +\infty] \), such that \( r(\delta) \to 0 \) as \( \delta \to 0 \) and
\[
\forall G \in \mathbb{R}^{3 \times 3} : \sup_{h>0} \sup_{x \in \Omega} |W^h(x, Id + G) - Q^h(x, G)| \leq |G|^2 r(|G|),
\]
where \( Q^h(x, \cdot) \) is a quadratic form given in Definition 2.1.

Notice that \( Q^h \) can be written as the pointwise limit
\[
(x, G) \to Q^h(x, G) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} W^h(x, Id + \varepsilon G),
\]
and therefore inherits the measurability properties of \( W^h \).

**Lemma 2.3.** Let \((W^h)_{h>0}\) be as in Definition 2.2 and let \((Q^h)_{h>0}\) be the quadratic form associated to \( W^h \) through the expansion (W4). Then
\[
\text{(Q1) for all } h > 0 \text{ and almost all } x \in \Omega \text{ the map } Q^h(x, \cdot) \text{ is quadratic and satisfies}
\]
\[
\eta_1 |\text{sym } G|^2 \leq Q^h(x, G) = Q^h(x, \text{sym } G) \leq \eta_2 |\text{sym } G|^2 \quad \text{for all } G \in \mathbb{R}^{3 \times 3}.
\]

**Proof.** (Q1) follows from (W2). \( \square \)

**Remark 1.** From (Q1) it follows
\[
|Q^h(x, G_1) - Q^h(x, G_2)| \leq \eta_2 |\text{sym } G_1 - \text{sym } G_2| \cdot |\text{sym } G_1 + \text{sym } G_2|, \quad \forall h > 0, G_1, G_2 \in \mathbb{R}^{3 \times 3}.
\]
Existing von Kármán plate models. In [FJM06], under the assumption that $W^h$ does not depend of $h$ and that is homogeneous, i.e., does not depend on $x \in \Omega$, the following limit energy was obtained (it was supposed that the sequence of minimizers $(y^h)_{h>0} \subset H^1(\Omega, \mathbb{R}^2)$ satisfies $\limsup_{h \to 0} I^h(y^h) < \infty$)

$$\frac{1}{2} \int_{\Omega} Q_2(\text{sym} \nabla' u + \frac{1}{2} \nabla' v \otimes \nabla' v)dx' + \frac{1}{24} \int_{\Omega} Q_2(\nabla'^2 v)dx'.$$

Here $u \in H^1(\omega; \mathbb{R}^2)$ is the limit in-plane displacement (scaled with $h^2$), $v \in H^2(\omega)$ is the limit vertical displacement (scaled with $h$), and

$$Q_2(A) = \min_{d \in \mathbb{R}^3} D^W(I)(t(A) + d \otimes e_3), \quad \text{for } A \in \mathbb{R}^{2 \times 2}.$$

The separation of stretching and bending energy in the limit functional does not appear when the material changes across the thickness (i.e., when energy density depends on the variable $x_3$), as well as when the material oscillates in the in-plane direction (see [NV13]). In the case of periodic oscillations in the in-plane direction the limit functional has the form

$$\frac{1}{2} \int_{\omega} Q^\gamma_{\text{hom}}(\text{sym} \nabla' u + \frac{1}{2} \nabla' v \otimes \nabla' v, -\nabla'^2 v)dx'.$$

where $\gamma = \lim_{h \to 0} \frac{h}{\varepsilon(h)}$ and $Q^\gamma_{\text{hom}} : \mathbb{R}^{2 \times 2}_{\text{sym}} \times \mathbb{R}^{2 \times 2}_{\text{sym}} \to \mathbb{R}$ is a positive definite quadratic form dependent on $\gamma$ and given in a minimization formulation over the cell $Y \times I$, $Y$ being the cell of the in-plane oscillations. For the form of the limit energy in the case analyzed here, see the expression (11) below.

Definition of convergence. With von Kárman model we associate the following triple (the limit deformation is rigid and the energy depends on the horizontal in-plane displacement $u$ and vertical displacement $v$, see Definition 2.4 and Theorem 2.6)

$$\left( \bar{R}, u, v \right) \in \text{SO}(3) \times A(\omega), \quad A(\omega) := \left\{ \left( u, v \right) : u \in H^1(\omega, \mathbb{R}^2), v \in H^2(\omega) \right\}.$$

The following definition and lemma can be found in [NV13]. The definition is changed in the way that we require less, i.e., strong convergence in $L^2$ instead of weak convergence in $H^1$. The role of the definition is to introduce the standard limit deformations (and kind of convergence of the minimizing sequence) in von Kárman regime.

Definition 2.4. We say that a sequence $(y^h)_{h>0} \subset L^2(\Omega, \mathbb{R}^2)$ converges to a triple $(\bar{R}, u, v) \in \text{SO}(3) \times L^2(\omega, \mathbb{R}^2) \times L^2(\omega)$, and write $y^h \rightharpoonup (\bar{R}, u, v)$, if there exist rotations $(\bar{R}^h)_{h>0}$ and functions $(u^h)_{h>0} \subset L^2(\omega, \mathbb{R}^2), (v^h)_{h>0} \subset L^2(\omega)$ such that

$$\lim_{h \to 0} \int_{I} y^h(x', x_3)dx_3 - \int_{\Omega} y^h dx = \begin{pmatrix} x' + h^2 w^h(x') \\ h v^h(x') \end{pmatrix},$$

for some $a \in \mathbb{R}^2, A \in \mathbb{R}^{2 \times 2}_{\text{skw}}$. The proof of the following lemma goes in the same way as the proof of [NV13] Lemma 2.1.
Lemma 2.5 (uniqueness). Let \((\tilde{R}, u, v), (\tilde{R}, \tilde{u}, \tilde{v}) \in SO(3) \times L^2(\omega, \mathbb{R}^2) \times L^2(\omega)\) and consider a sequence \((y^h)_{h>0}\) that converges to \((\tilde{R}, u, v)\). Then
\[
y^h \to (\tilde{R}, \tilde{u}, \tilde{v}) \iff \tilde{R} = \tilde{R} \text{ and } (u, v) \sim (\tilde{u}, \tilde{v}).
\]

We will now state the main result, just for the reader's convenience. Its proof is the direct consequence of Theorem 2.12, Theorem 2.13, Remark 4, Remark 6 and Proposition 2.10.

Theorem 2.6. Let \((W^h)_{h>0}\) satisfies the Definition 2.2 for some \(\eta_1, \eta_2, \eta > 0\). Let \((h_n)_{n \in \mathbb{N}}\) be an arbitrary sequence monotonically decreasing to zero. Then there exists a subsequence, still denoted by \((h_n)_{n \in \mathbb{N}}\), and a limit functional \(I^0 : \mathcal{A}(\omega) \to \mathbb{R}_0^+\) such that

(i) (Compactness). Let \((y^{h_n})_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) be a sequence with equibounded energy, that is \(\limsup_{n \to \infty} I^{h_n}(y^{h_n}) < \infty\). Then there exists \((\tilde{R}, u, v) \in SO(3) \times \mathcal{A}(\omega)\) such that
\[
y^{h_n} \to (\tilde{R}, u, v) \text{ up to a subsequence.}
\]

(ii) (Lower bound). Let \((y^{h_n})_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) be a sequence with equibounded energy. Assume that \(y^{h_n} \to (\tilde{R}, u, v)\). Then
\[
\liminf_{n \to \infty} I^{h_n}(y^{h_n}) \geq I^0(u, v).
\]

(iii) (recovery sequence). For every \((\tilde{R}, u, v) \in SO(3) \times \mathcal{A}(\omega)\) there exists a sequence \((y^{h_n})_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) with
\[
y^{h_n} \to (\tilde{R}, u, v) \quad \text{and} \quad \lim_{n \to \infty} I^{h_n}(y^{h_n}) = I^0(u, v).
\]

The limit functional \(I^0\) is given by [11] below and the energy density \(Q : \omega \times \mathbb{R}^{2 \times 2}_{\text{sym}} \times \mathbb{R}^{2 \times 2}_{\text{sym}} \to \mathbb{R}\) satisfies the property \((Q'1)\) below and is given by the asymptotic formula [1].

Statement of the main claims. The following two definitions are analogous to \(\Gamma\)-lim inf and \(\Gamma\)-lim sup. They are used to obtain that the Assumption 2.8 is satisfied on a subsequence (see Lemma 2.7). We define for the sequence \((h_n)_{n \in \mathbb{N}}\) which monotonically decreases to zero and arbitrary \(A \subset \omega\) open and \(M \in L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}),\)

\[
K^-(h_n)(M, A) = \inf \left\{ \liminf_{n \to \infty} \int_{A \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi^{h_n}) \; dx : \right. \\
\left. (\psi^{h_n}_1, \psi^{h_n}_2, h_n \psi^{h_n}_3) \to 0 \text{ strongly in } L^2(A \times I, \mathbb{R}^3) \right\}
\]

\[
= \sup_{\mathcal{N}(0) \cup \mathcal{A}(0)} \liminf_{n \to \infty} \inf_{\psi \in H^1(A \times I, \mathbb{R}^3)} \int_{A \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi) \; dx,
\]

\[
K^+(h_n)(M, A) = \inf \left\{ \limsup_{n \to \infty} \int_{A \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi^{h_n}) \; dx : \\
(\psi^{h_n}_1, \psi^{h_n}_2, h_n \psi^{h_n}_3) \to 0 \text{ strongly in } L^2(A \times I, \mathbb{R}^3) \right\}
\]

\[
= \sup_{\mathcal{N}(0) \cup \mathcal{A}(0)} \limsup_{n \to \infty} \inf_{\psi \in H^1(A \times I, \mathbb{R}^3)} \int_{A \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi) \; dx.
\]

By \(\mathcal{N}(0)\) we have denoted the family of all neighbourhoods of zero in the strong \(L^2\) topology.
Remark 2. Since the above expressions are monotonically decreasing in $N(0)$ it is enough to take the supremum on the monotone sequence of neighbourhoods that shrinks to \{0\}, e.g., the sequence of (open or closed) balls of radius $r$, when $r \to 0$.

Remark 3. By using standard diagonalization argument it can be shown that for any $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero and any $A \subset \omega$ open and $M \in L^2(\Omega, \mathbb{R}^{2\times 2})$ it holds

$$K_{(h_n)}^-(M, A) = \min \left\{ \lim_{n \to \infty} \int_{A \times I} Q^{h_n} \left( x, \nu(M) + \nabla h_n \psi_{h_n} \right) \, dx : \right.$$ 

$$(\psi_{h_n}^1, \psi_{h_n}^2, h_n \psi_{h_n}^3) \to 0 \text{ strongly in } L^2(A \times I, \mathbb{R}^3) \},$$

$$K_{(h_n)}^+(M, A) = \min \left\{ \lim_{n \to \infty} \int_{A \times I} Q^{h_n} \left( x, \nu(M) + \nabla h_n \psi_{h_n} \right) \, dx : \right.$$ 

$$(\psi_{h_n}^1, \psi_{h_n}^2, h_n \psi_{h_n}^3) \to 0 \text{ strongly in } L^2(A \times I, \mathbb{R}^3) \}.$$

Let $D$ denote a countable family of open subsets of $\omega$ which is dense in the family of all open subsets of $\omega$ (see Definition A.9) and such that every $D \in \mathcal{D}$ is of class $C^{1,1}$. An existence of such family can be easily verified. The following lemma uses the standard diagonalization argument and Lemma 3.4. Since the proof is easy we will give it immediately here, although in this way we are violating a bit the structure of the paper.

Lemma 2.7. For every sequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero there exists a subsequence, still denoted by $(h_n)_{n \in \mathbb{N}}$, such that

$$K_{(h_n)}^+(M, D) = K_{(h_n)}^-(M, D), \quad \forall M \in L^2(\Omega, \mathbb{R}^{2\times 2}), \quad \forall D \in \mathcal{D}.$$

Proof. Take a countable family $\{M_j\}_{j \in \mathbb{N}} \subset L^2(\Omega, \mathbb{R}^{2\times 2})$ which is dense in $L^2(\Omega, \mathbb{R}^{2\times 2})$. By a diagonalizing argument it is not difficult to construct a subsequence $(h_n)$ monotonically decreasing to zero such that

$$K_{(h_n)}^+(M_j, D) = K_{(h_n)}^-(M_j, D), \quad \forall j \in \mathbb{N}, \quad \forall D \in \mathcal{D}.$$

From Lemma 3.4 and density we have the claim. \hfill \qedsymbol

We now give the following assumption.

Assumption 2.8. For a sequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero we suppose that for every $D \in \mathcal{D}$ and every $M \in L^2(\Omega, \mathbb{R}^{2\times 2})$ there exists $K(M, D)$ such that we have

$$K_{(h_n)}^+(M, D) = K_{(h_n)}^-(M, D) =: K(M, D).$$

Remark 4. The Assumption 2.8 is the assumption supposed in Theorem 2.12 and Theorem 2.13. Lemma 2.7 states that the assumption (and thus the mentioned theorems) are satisfied on a subsequence.

The following lemma is easy to prove by a contradiction (see the proof of [Bra02, Proposition 1.44]).

Lemma 2.9. Suppose that for a given sequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero the following holds: for every $M \in L^2(\Omega, \mathbb{R}^{2\times 2})$ and every $D \in \mathcal{D}$ there exists $K(M, D)$ such that every subsequence $(h_{n(k)})_{k \in \mathbb{N}}$ monotonically decreasing to zero has a subsequence, still denoted by $(h_{n(k)})_{k \in \mathbb{N}}$, which satisfies

$$K(M, D) = K_{(h_{n(k)})_{k \in \mathbb{N}}}^-(M, D).$$

Then the Assumption 2.8 is satisfied.
We introduce the space \( \mathcal{S}_{vK}(\Omega) \subset L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}) \) of matrix fields which appear as limit strains in von Kármán model
\[
\mathcal{S}_{vK}(\Omega) = \{ M_1 + x_3 M_2 : M_1, M_2 \in L^2(\omega, \mathbb{R}^{2 \times 2}_{\text{sym}}) \}.
\]

**Remark 5.** Starting from Lemma 2.7 and Assumption 2.8 we could state the results, only restricting ourselves on the space \( \mathcal{S}_{vK}(\Omega) \) instead of \( L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}) \). We refrained ourselves from doing so for the sake of generality, when it was meaningful.

The proofs of the following claims are put in Section 3. The next proposition is one of the key claims of the paper. It corresponds to standard claim of integral representation of \( \Gamma \)-limit. Here we are able to specify that \( Q \) is quadratic with some additional properties.

**Proposition 2.10.** Let \((h_n)_{n \in \mathbb{N}}\) be a sequence for which the Assumption 2.8 is satisfied. Then there exists a function \( Q : \omega \times \mathbb{R}^{2 \times 2}_{\text{sym}} \times \mathbb{R}^{2 \times 2}_{\text{sym}} \to \mathbb{R} \) (dependent on this sequence) such that for every \( A \subset \omega \) open and every \( M \in \mathcal{S}_{vK}(\Omega) \) of the form
\[
M = M_1 + x_3 M_2, \text{ for some } M_1, M_2 \in L^2(\omega, \mathbb{R}^{2 \times 2}_{\text{sym}}),
\]
we have
\[
(10) \quad K(M, A) = \int_A Q(x', M_1(x'), M_2(x')) \, dx'.
\]

Moreover, \( Q \) satisfies the following property
\[
(\text{Q'1}) \quad \text{for almost all } x' \in \omega \text{ the map } Q(x', \cdot, \cdot) \text{ is a quadratic form and satisfies}
\]
\[
\eta_1 \left( |G_1|^2 + |G_2|^2 \right) \leq Q(x', G_1, G_2) \leq \eta_2 \left( |G_1|^2 + |G_2|^2 \right) \quad \text{for all } G_1, G_2 \in \mathbb{R}^{2 \times 2}_{\text{sym}}.
\]

The following corollary gives the alternative to the Assumption 2.8.

**Corollary 2.11.** Let \((h_n)_{n \in \mathbb{N}}\) be a sequence monotonically decreasing to zero. Assume that for almost every \( x' \in \omega \) there exists a sequence \((r_{x'}^m)_{m \in \mathbb{N}}\) converging to zero and numbers \( K\left(M, B(x', r_{x'}^m)\right)\) such that
\[
K\left(M, B(x', r_{x'}^m)\right) = K_{(h_n)}^\pm \left(M, B(x', r_{x'}^m)\right),
\]
\[
\forall m \in \mathbb{N}, M = M_1 + x_3 M_2; M_1, M_2 \in \mathbb{R}^{2 \times 2}_{\text{sym}}.
\]

Then for every \( M \in \mathcal{S}_{vK}(\Omega), A \subset \omega \) open there exists \( K(M, A) \) such that the following property holds
\[
K(M, A) = K_{(h_n)}^\pm (M, A) = K_{(h_n)}^\pm (M, A).
\]

Denote by \( I^0 : \mathcal{A}(\omega) \to \mathbb{R}^+_0 \) the functional
\[
(11) \quad I^0(u, v) = \int_\omega Q(x', \nabla u + \frac{1}{2} \nabla v \otimes \nabla v, -\nabla^2 v) \, dx'.
\]

Notice that for \((u_1, v_1), (u_2, v_2) \in \mathcal{A}(\omega)\) with \((u_1, v_1) \sim (u_2, v_2)\) we have \( I^0(u_1, v_1) = I^0(u_2, v_2)\).

The following two theorems are analogous to standard lower and upper bound and are already stated as the part of Theorem 2.6.

**Theorem 2.12.** Let \((h_n)_{n \in \mathbb{N}}\) be a sequence monotonically decreasing to zero for which the Assumption 2.8 is satisfied.
(i) (Compactness). Let \((y^h_n)_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) be a sequence with equibounded energy, that is \(\limsup_{n \to \infty} I^h_n(y^h_n) < \infty\). Then there exists \((R, u, v) \in \text{SO}(3) \times \mathcal{A}(\omega)\) such that \(y^h_n \to (R, u, v)\) up to a subsequence.

(ii) (Lower bound). Let \((y^h_n)_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) be a sequence with equibounded energy. Assume that \(y^h_n \to (R, u, v)\). Then
\[
\liminf_{n \to \infty} I^h_n(y^h_n) \geq I^0(u, v).
\]

**Theorem 2.13.** (recovery sequence). Let \((h_n)_{n \in \mathbb{N}}\) be a sequence monotonically decreasing to zero for which the Assumption 2.8 is satisfied. Then for every \((R, u, v) \in \text{SO}(3) \times \mathcal{A}(\omega)\) there exists a subsequence, still denoted by \((h_n)_{n \in \mathbb{N}}\), such that there exists a sequence \((y^h_n)_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)\) with
\[
y^h_n \to (R, u, v) \quad \text{and} \quad \lim_{n \to \infty} I^h_n(y^h_n) = I^0(u, v).
\]

**Remark 6.** Although Theorem 2.13 is stated in the form such that the subsequence can depend on \((R, u, v)\), by separability and diagonal argument, we conclude that there exists a subsequence such that the claim of Theorem 2.13 holds for every \((R, u, v) \in \text{SO}(3) \times \mathcal{A}(\omega)\). Nevertheless, it can be easily checked that Theorem 2.12 and Theorem 2.13 do imply in their form the convergence of minimizers to the minimizer of the limit functional (if we also add to them appropriate external loads, see [FJM06], on every subsequence of the sequence \((h_n)_{n \in \mathbb{N}}\), where they are converging. Thus, we can even the statement of Theorem 2.13 in its form look at as upper bound statement.

**The strategy of the proofs.** The proofs use some standard approach of \(\Gamma\)-convergence adapted to this situation with specific compactness result. It is easy to see that as a consequence of objectivity we have
\[
W^h(x, \nabla_h y^h) = W^h(x, Id + h^2 E^h),
\]
whenever \(\det \nabla_h y^h > 0\). Here \((E^h)_{h>0}\) is a sequence of limiting scaled strains given by
\[
E^h = \frac{(\nabla_h y^h)^t \nabla_h y^h - Id}{h^2}.
\]
It can be shown that if \(\limsup_{h \to 0} I^h(y^h) < \infty\), then the sequence \((E^h)_{h>0}\) is bounded in \(L^2\), after eliminating small sets (and also that \(\det \nabla_h y^h \leq 0\) on a small set). The essential part of the proof of Theorem 2.12 consists in writing the sequence of limiting scaled strains in the form (up to term which converges to zero strongly in \(L^2\) and on a “large set”)
\[
E^h \approx \text{sym } \nabla u - \frac{1}{2} \nabla v \otimes \nabla v - x_3 \nabla^2 v + \text{sym } \nabla h \psi^h,
\]
where \((\psi^h_1, \psi^h_2, h \psi^h_3) \to 0\) strongly in \(L^2\) and \((\text{sym } \nabla h \psi^h)_{h>0}\) is bounded in \(L^2\). In order to do that we use the result [NV13 Proposition 3.1] which identifies the behaviour of deformations of energy of order \(h^4\) (von Kármán regime). This identification enables us to define the limiting energy, where we relax the energy with respect to the sequence \(\nabla_h \psi^h\). This naturally imposes that in the asymptotic formula we separate fixed \(L^2\) field, that represents the limit strain, and variable relaxation sequence \(\nabla_h \psi^h\) (since the density satisfies the property (Q1) defining the relaxation sequence with \(\nabla_h \psi^h\) is the same as defining it with \(\text{sym } \nabla_h \psi^h\)). That is how we come to the definition of the functional \(K(\cdot, \cdot)\). This separation is not done in standard \(\Gamma\)-convergence techniques, but here it is also helpful for proving the properties of the functional \(K(\cdot, \cdot)\), i.e., for establishing its representation, given in Proposition 2.10.
In order to make the lower bound we need to use the truncation argument, but we have to preserve the structure of symmetrized scaled gradients. This is done by using a bit modified Griso’s decomposition which says that (see Proposition 3.3)

\[
\psi^h \approx \begin{pmatrix}
0 \\
0 \\
\frac{\hat{\psi}^h}{h}
\end{pmatrix} - x_3 \begin{pmatrix}
\partial_1 \phi^h \\
\partial_2 \phi^h \\
0
\end{pmatrix} + \bar{\psi}^h,
\]

i.e.,

\[
\text{sym } \nabla_h \psi^h = -x_3 t (\nabla^2 \phi^h) + \text{sym } \nabla_h \bar{\psi}^h,
\]

up to a term converging to zero strongly in \( L^2 \). Here we have that \( \phi^h \to 0 \) (\( \phi^h \) are only functions of in-plane variables) and \( \bar{\psi}^h \to 0 \) strongly in \( L^2 \) and \( (\nabla \bar{\psi}^h)_{h>0} \) are bounded in \( L^2 \) (in fact, a slight modification of [NV13, Proposition 3.1], that is done directly in the proof of lower bound, gives us directly the strain in this form, i.e., in expression (12), \( \text{sym } \nabla \bar{\psi}^h \) is replaced by (14)). We can then replace both of these sequences by equi-integrable sequences recalling some standard results from the literature (see [FMP98, BF02, BZ07]). The adaptation of these standard results to this situation is analyzed in the Appendix. This replacement is crucial and enables us to make the lower bound.

The proof of Proposition 2.10 follows, in its basic ideas, the standard approach for obtaining integral representation (and for proving quadraticity). Of course, we use the convenience that we are dealing with quadratic functionals. Instead of using De Giorgi slicing argument, we again use the equi-integrability property of the minimizing sequence (for this we now need Griso’s decomposition).

In the proof of the upper bound we use the asymptotic formula and we replace the relaxation field with the field of scaled gradients bounded in \( L^\infty \). This can be done by adaptation of Lusin’s type approximation already exploited in the derivation of lower dimensional models (see [FJM02]). The basic fact used here is that we can replace the equi-integrable sequence by \( L^\infty \) sequence, and that the estimate of the error is uniform with respect to the index of the sequence.

To summarize: we prove lower and upper bound by using only the equi-integrability of the minimizing sequence; Griso’s decomposition is used to write asymptotic formula in different way, i.e., to obtain expression (14), while [NV13, Proposition 3.1] is used to decompose the sequence of strains in the form given by the expressions (12) and (14).

### 3. Proofs

#### 3.1. Characterization of the symmetrized scaled gradients.

The following theorem is proved in [Gri05]. We use it to obtain the characterization of the sequence \( \psi^h \in H^1(\Omega, \mathbb{R}^3) \) which satisfies the property that \( (\text{sym } \nabla_h \psi^h)_{h>0} \) is bounded in \( L^2 \) and \( (\psi_1^h, \psi_2^h, \psi_3^h) \to 0 \), strongly in \( L^2 \) (see Proposition 3.3). In the claims below when we put \( C(A) \), it means that the constant depends on the domain \( A \), but not on functions and variable \( h \).

**Theorem 3.1.** Let \( A \subset \omega \) with Lipschitz boundary and \( \psi \in H^1(A \times I, \mathbb{R}^3) \) and \( h > 0 \). Then we have the following decomposition

\[
\psi(x) = \hat{\psi}(x') + r(x') \wedge x_3 e_3 + \bar{\psi}(x) = \begin{cases}
\hat{\psi}_1(x') + r_2(x') x_3 + \bar{\psi}_1(x) \\
\hat{\psi}_2(x') - r_1(x') x_3 + \bar{\psi}_2(x) \\
\hat{\psi}_3(x') + \bar{\psi}_3(x)
\end{cases},
\]

where \( \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3 \) are functions of \( x' \) and \( r_1, r_2 \) are functions of \( x \) and \( x_3 \).


Thus, from Korn’s inequality it follows

\[ \| \text{sym} \nabla_h (\hat{\psi} + r \wedge x_3 e_3) \|_{L^2}^2 + \| \nabla_h \hat{\psi} \|_{L^2}^2 + \frac{1}{r^2} \| \hat{\psi} \|_{L^2}^2 \leq C(A) \| \text{sym} \nabla_h \psi \|_{L^2}^2. \]

Remark 7. Notice that

\[ \| \text{sym} \nabla_h (\hat{\psi} + r \wedge x_3 e_3) \|_{L^2(A \times I)}^2 = \]

\[ \| \text{sym} \nabla' (\hat{\psi}_1, \hat{\psi}_2) \|_{L^2(A)}^2 + \frac{1}{r^2} \| \text{sym} \nabla' (r_2, -r_1) \|_{L^2(A)}^2 \]

\[ + \frac{1}{r^2} \| \partial_1 (h \hat{\psi}_3) + r_2 \|_{L^2(A)}^2 + \frac{1}{r^2} \| \partial_2 (h \hat{\psi}_3) - r_1 \|_{L^2(A)}^2. \]

Thus, from Korn’s inequality it follows

\[ \|(\hat{\psi}_1, \hat{\psi}_2, h \hat{\psi}_3)\|_{H^1(A)}^2 + \|(r_1, r_2)\|_{H^1(A)}^2 + \frac{1}{r^2} \| \partial_1 (h \hat{\psi}_3) + r_2 \|_{L^2(A)}^2 \]

\[ + \frac{1}{r^2} \| \partial_2 (h \hat{\psi}_3) - r_1 \|_{L^2(A)}^2 \]

\[ \leq C(A) \left( \| \text{sym} \nabla_h (\hat{\psi} + r \wedge x_3 e_3) \|_{L^2(A \times I)}^2 + \| r \|_{L^2(A)}^2 + \| (\hat{\psi}_1, \hat{\psi}_2, h \hat{\psi}_3) \|_{L^2(A \times I)}^2 \right) \]

\[ \leq C(A) \left( \| \text{sym} \nabla_h (\hat{\psi} + r \wedge x_3 e_3) \|_{L^2(A \times I)}^2 + \| (\hat{\psi}_1, \hat{\psi}_2, h \hat{\psi}_3) \|_{L^2(A \times I)}^2 \right). \]

The following lemma is crucial for proving Proposition 3.3.3

Lemma 3.2. Let \( A \subseteq \omega \) with \( C^{1,1} \) boundary and \( h > 0 \). If \( r \in H^1(A, \mathbb{R}^2) \) and \( \hat{\psi} \in H^1(A, \mathbb{R}^3) \) are such that \( \int_{A_i} \hat{\psi}_3 \, dx = 0 \), for every connected component \( A_i \) of \( A \) (see Lemma A.1.2), then there exists \( \varphi \in H^2(A) \) and \( w \in H^1(A) \) such that \( \hat{\psi}_3 = \frac{\partial}{\partial n} + w \) and

\[ \| \varphi \|_{H^2(A)}^2 + \| (\hat{\psi}_1, \hat{\psi}_2) \|_{H^1(A)}^2 + \| r \|_{H^1(\omega)}^2 + \| w \|_{H^1(A)}^2 + \frac{1}{r^2} \| \partial_1 \varphi + r_2 \|_{L^2(A)}^2 \]

\[ + \frac{1}{r^2} \| \partial_2 \varphi - r_1 \|_{L^2(A)}^2 \leq C(A) \left( \| \text{sym} \nabla_h (\hat{\psi} + r \wedge x_3 e_3) \|_{L^2(A \times I)}^2 + \| r \|_{L^2(A)}^2 + \| (\hat{\psi}_1, \hat{\psi}_2, h \hat{\psi}_3) \|_{L^2(A \times I)}^2 \right). \]

Proof. We do the regularization of \( r \) in the similar way as in [NV13] Proposition 3.1 and [HV14] Lemma 3.8. We look for the solution of the problem

\[ \min_{\varphi \in H^1(A)} \int_A \| \nabla' \varphi + (r_2, -r_1) \|_{L^2(A \times I)}^2 \]

The associated Euler-Lagrange equation of (20) reads

\[ \begin{cases} -\Delta' \varphi = \nabla' \cdot (r_2, -r_1) & \text{in } A \\ \partial_n \varphi = -(r_2, -r_1) \cdot \nu & \text{on } \partial A. \end{cases} \]

Since \( \nabla' \cdot (r_2, -r_1) \in L^2 \), we obtain by standard regularity estimates that \( \varphi \in H^2(A) \) and \( \| \varphi \|_{H^2(A)} \leq C(A) \| r \|_{H^1(A)} \), where we need \( C^{1,1} \) regularity of \( \partial A \). The claim follows from
components of \( \hat{\psi} \) we can demand additionally that

\[ \text{Remark 8.} \] If we assume that \( \hat{\psi} = 0 \), \( r = 0 \) on \( \partial A \times I \) (without the assumption that \( \int_{A_i} \hat{\psi}_3 \, dx = 0 \), for every connected component \( A_i \) of \( A \)) the claim of Lemma \( 3.2 \) holds and we can demand additionally that \( \varphi \in H^2(A) \), \( \varphi = 0 \) on \( \partial A \), \( w \in H^1_0(A) \). Using again Korn’s inequality with boundary conditions we can omit \( \| r \|_{L^2} \) and \( \| (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3) \|_{L^2} \) on the right hand side of (19). To adapt the proof of Lemma \( 3.2 \) instead of solving problem (20), we need to solve the problem

\[ \min_{\varphi \in H^1_0(A)} \int_A |\nabla \varphi + (r_2, -r_1)|^2 \, dx'. \]

The following proposition gives the characterization of the symmetrized scaled gradients we will work with.

**Proposition 3.3.** Let \( A \subset \omega \) with \( C^{1,1} \) boundary. Denote by \( \{A_i\}_{i=1,\ldots,k} \) the connected components of \( A \).

(a) Let \((\psi^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)\) be such that

\[ (\psi^h_1, \psi^h_2, \psi^h_3) \to 0, \text{ strongly in } L^2, \quad \forall h, i \int_{A_i} \psi^h_3 = 0, \]

\[ \lim_{h \to 0} \| \text{sym } \nabla_h \psi^h \|_{L^2(A)} \leq M < \infty. \]

Then there exist \((\varphi^h)_{h>0} \subset H^2(A), (\hat{\psi}^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)\) such that

\[ \text{sym } \nabla_h \psi^h = -x_{36}(\nabla^2 \varphi^h) + \text{sym } \nabla_h \hat{\psi}^h + o^h, \]

where \( o^h \in L^2(A \times I, \mathbb{R}^3 \times \mathbb{R}^3) \) is such that \( o^h \to 0 \), strongly in \( L^2 \), and the following properties hold

\[ \lim_{h \to 0} \left( \| \varphi^h \|_{H^1(A)} + \| \hat{\psi}^h \|_{L^2(A \times I)} \right) = 0, \]

\[ \limsup_{h \to 0} \left( \| \varphi^h \|_{H^2(A)} + \| \nabla_h \hat{\psi}^h \|_{L^2(A \times I)} \right) \leq C(A)M. \]

(b) For every \((\varphi^h)_{h>0} \subset H^2(A), (\hat{\psi}^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)\) such that

\[ \lim_{h \to 0} \left( \| \varphi^h \|_{H^1(A)} + \| \hat{\psi}^h \|_{L^2(A \times I)} \right) = 0, \]

\[ \limsup_{h \to 0} \left( \| \varphi^h \|_{H^2(A)} + \| \nabla_h \hat{\psi}^h \|_{L^2(A)} \right) \leq M, \]

there exists a sequence \((\psi^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)\) such that

\[ \text{sym } \nabla_h \psi^h = -x_{36}(\nabla^2 \varphi^h) + \text{sym } \nabla_h \hat{\psi}^h, \]

\[ (\psi^h_1, \psi^h_2, \psi^h_3) \to 0 \text{ strongly in } L^2, \]

\[ \limsup_{h \to 0} \| \text{sym } \nabla_h \psi^h \|_{L^2(A)} \leq 2M. \]
Proof. The proof follows immediately from Theorem 3.1 and Lemma 3.2. Namely, if we use the decomposition from Theorem 3.1 and Lemma 3.2 we obtain

\begin{equation}
\psi^h = \hat{\psi}^h + r^h \wedge x_3 e_3 + \overline{\psi}^h
\end{equation}

From the expressions (15) and (16) it follows (\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3) \to 0, (r^h_1, r^h_2) \to 0, \overline{\psi}^h \to 0 strongly in \(L^2\). This implies \(\varphi^h \to 0\) strongly in \(L^2\). Since it holds for every \(i\), \(\int_A \hat{\psi}_3 \, dx = 0\) (see (15)), from (16) and Lemma 3.2 we have that \(\limsup_{h \to 0} \|w^h\|_{H^2(A)} \leq C(A)M\) and since \((\varphi^h)_{h>0}\) is bounded in \(H^2\) we deduce by compactness that \(\varphi^h \to 0\) strongly in \(H^1\). Thus, we can find \((\tilde{w}^h)_{h>0} \subset H^2(A)\) such that

\begin{equation}
\lim_{h \to 0} \|w^h - \tilde{w}^h\|_{L^2(A)} = 0, \ \limsup_{h \to 0} \|w^h\|_{H^2(A)} \leq C(A)M, \ \lim h\|\tilde{w}^h\|_{H^2(A)} = 0.
\end{equation}

This can be done by mollifying \(w^h\) with the mollifiers on the scale \(\varepsilon^h \to 0, \varepsilon^h \gg h\). From (30) we have

\begin{equation}
\psi^h = \left( \begin{array}{c}
\hat{\psi}^h \\
\hat{\psi}_2 \\
0
\end{array} \right) + \left( \begin{array}{c}
0 \\
0 \\
\varphi^h_h + \tilde{w}^h
\end{array} \right) - x_3 \left( \begin{array}{c}
\partial_1 \varphi^h \\
\partial_2 \varphi^h \\
0
\end{array} \right) + x_3 \left( \begin{array}{c}
\partial_1 \varphi^h + r^h_2 \\
\partial_2 \varphi^h - r^h_1 \\
0
\end{array} \right)
\end{equation}

Now the claim follows from Lemma 3.2 and (31) by defining

\begin{align*}
\tilde{\psi}^h &= \left( \begin{array}{c}
\hat{\psi}^h \\
\hat{\psi}_2 \\
0
\end{array} \right) + x_3 \left( \begin{array}{c}
\partial_1 \varphi^h + r^h_2 \\
\partial_2 \varphi^h - r^h_1 \\
0
\end{array} \right) + h x_3 \left( \begin{array}{c}
\partial_1 \tilde{w}^h \\
\partial_2 \tilde{w}^h \\
0
\end{array} \right) + \left( \begin{array}{c}
0 \\
0 \\
w^h - \tilde{w}^h
\end{array} \right) + \overline{\psi}^h, \\
o^h &= -h x_3 \left( \nabla^2 \tilde{w}^h \right),
\end{align*}

after using the identity

\[
sym \nabla_h \left( \begin{array}{c}
0 \\
0 \\
w^h
\end{array} \right) = sym \nabla_h \left( \begin{array}{c}
h x_3 \partial_1 \tilde{w}^h \\
h x_3 \partial_2 \tilde{w}^h \\
0
\end{array} \right) - h x_3 \ell \left( \nabla^2 \tilde{w} \right).
\]

The second part of proposition is direct by defining

\[
\psi^h = \tilde{\psi}^h + \left( \begin{array}{c}
0 \\
0 \\
\varphi^h_h
\end{array} \right) - x_3 \left( \begin{array}{c}
\partial_1 \varphi^h \\
\partial_2 \varphi^h \\
0
\end{array} \right).
\]

\[\square\]

3.2. Properties of \(K(\cdot, \cdot)\). Here we want to establish some important properties of \(K(\cdot, \cdot)\) (see Lemma 3.7) which will help us to prove Proposition 2.10. We emphasize the fact that all these properties are simple consequence of Lemma 3.4 and Lemma 3.5.
Lemma 3.4 (continuity in \( M \)). There exists a constant \( C > 0 \) dependent only on \( \eta_1, \eta_2 \) such that for every sequence \( (h_n)_{n \in \mathbb{N}} \) monotonically decreasing to zero and \( A \subset \omega \) open set we have

\[
\left| K^\pm\left( M_1, A \right) - K^\pm\left( M_2, A \right) \right| \leq C\| M_1 - M_2 \|_{L^2} \left( \| M_1 \|_{L^2} + \| M_2 \|_{L^2} \right),
\]

\( \forall M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2 \times 2}) \).

**Proof.** We will prove the claim only for \( K^-(h_n) \). For fixed \( M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2 \times 2}) \) take an arbitrary \( r, h > 0 \) and \( \psi^{r,h_n}_\alpha \in H^1(A \times I, \mathbb{R}^3) \) that satisfy for \( \alpha = 1, 2 \)

\[
\int_\Omega Q^{h_n} \left( x, \iota(M_\alpha) + \nabla_{h_n} \psi^{r,h_n}_\alpha \right) \, dx \leq \inf_{\psi \in H^1(A \times I, \mathbb{R}^3)} \int_{A \times I} Q^{h_n} \left( x, \iota(M_\alpha) + \nabla_{h_n} \psi \right) \, dx + h_n,
\]

\[
\left\| (\psi^{r,h_n}_\alpha, \eta^{r,h_n}_\alpha, h_n \psi^{r,h_n}_\alpha) \right\|_{L^2} \leq r.
\]

We want to prove that for every \( r > 0 \) we have

\[
\left| \int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_\alpha \right) \, dx - \int_{A \times I} Q^{h_n} \left( x, \iota(M_2) + \nabla_{h_n} \psi^{r,h_n}_\alpha \right) \, dx \right| \leq C\| M_1 - M_2 \|_{L^2} \left( \| M_1 \|_{L^2} + \| M_2 \|_{L^2} \right) + h_n.
\]

From that we can easily obtain (33).

Let us prove (35). From (34) and (Q1), by testing with zero function, we can assume for \( \alpha = 1, 2 \)

\[
\eta_1 \| M_\alpha + \text{sym} \nabla_{h_n} \psi^{r,h_n}_\alpha \|_{L^2}^2 \leq \int_{A \times I} Q^{h_n} \left( x, \iota(M_\alpha) + \nabla_{h_n} \psi^{r,h_n}_\alpha \right) \, dx \leq \eta_2 \| M_\alpha \|_{L^2}^2.
\]

From this we have for \( \alpha = 1, 2 \)

\[
\| \text{sym} \nabla_{h_n} \psi^{r,h_n}_\alpha \|_{L^2}^2 \leq C(\eta_1, \eta_2) \| M_\alpha \|_{L^2}^2.
\]

Without any loss of generality we can also assume that

\[
\int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_1 \right) \, dx \geq \int_{A \times I} Q^{h_n} \left( x, \iota(M_2) + \nabla_{h_n} \psi^{r,h_n}_2 \right) \, dx.
\]

We have

\[
\int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_1 \right) \, dx - \int_{A \times I} Q^{h_n} \left( x, \iota(M_2) + \nabla_{h_n} \psi^{r,h_n}_2 \right) \, dx = \int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_1 \right) \, dx - \int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_2 \right) \, dx
\]

\[
= \int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_1 \right) \, dx - \int_{A \times I} Q^{h_n} \left( x, \iota(M_2) + \nabla_{h_n} \psi^{r,h_n}_2 \right) \, dx
\]

\[
+ \int_{A \times I} Q^{h_n} \left( x, \iota(M_1) + \nabla_{h_n} \psi^{r,h_n}_1 \right) \, dx - \int_{A \times I} Q^{h_n} \left( x, \iota(M_2) + \nabla_{h_n} \psi^{r,h_n}_2 \right) \, dx \leq h_n + C(\eta_1, \eta_2)\| M_1 - M_2 \|_{L^2} \left( \| M_1 \|_{L^2} + \| M_2 \|_{L^2} \right).
\]

□

The following lemma will be working lemma for establishing the properties of \( K(\cdot, \cdot) \).
Lemma 3.5. Let for \((h_n)_{n \in \mathbb{N}}\) monotonically decreasing to zero Assumption 2.8 is satisfied. Suppose that for \(M \in L^2(\Omega, \mathbb{R}^{2 \times 2})\) and \(D \subset \Omega\) with \(C^{1,1}\) boundary we have 
\[
K(M, D) = \lim_{n \to \infty} \int_{D \times I} Q^{h_n}(x, t(M) + \nabla h_n \psi^{h_n}) \, dx,
\]
for some \((\psi^{h_n})_{n \in \mathbb{N}}\) such that \((\psi_1^{h_n}, \psi_2^{h_n}, h_n \psi_3^{h_n}) \to 0 \text{ strongly in } L^2\). Then there exists a subsequence \((h_n)_{k \in \mathbb{N}}\) and \((\vartheta_k)_{k \in \mathbb{N}} \subset H^1(D \times I, \mathbb{R}^3)\) such that 
(a) \((\vartheta_{k,1}, \vartheta_{k,2}, h_n(\vartheta_{k,3})) \to 0 \text{ strongly in } L^2\),
(b) \(|\text{sym } \nabla h_n(\vartheta_{k,3})^2|\), \(k \in \mathbb{N}\) is equi-integrable,
\[
\text{sym } \nabla h_n(\vartheta_{k}) = -x_3 \ell(\nabla^2 \varphi_k) + \text{sym } \nabla h_n(\tilde{\vartheta}_k),
\]
where \((|\nabla^2 \varphi_k|)_{k \in \mathbb{N}}\) and \((|\nabla h_n(\tilde{\vartheta}_k)|^2)_{k \in \mathbb{N}}\) are equi-integrable and \(\varphi_k \to 0 \text{ strongly in } H^1\) and \(\tilde{\vartheta}_k \to 0 \text{ strongly in } L^2\). Also the following holds 
\[
\limsup_{k \to \infty} \left(\|\varphi_k\|_{H^2(D)} + \|\nabla h_n(\tilde{\vartheta}_k)^2\|_{L^2(D)}\right) \leq C(D) \left(\eta_2 \|M\|_{L^2}^2 + 1\right).
\]
(c) 
\[
K(M, D) = \lim_{k \to \infty} \int_{D \times I} Q^{h_n(\vartheta_{k,3})}(x, t(M) + \nabla h_n(\vartheta_{k,3}) \vartheta_{k}) \, dx.
\]
Moreover, one can additionally assume that for each \(k \in \mathbb{N}\) we have \(\vartheta_{k} = 0\) on \(\partial D \times I\), i.e., \(\varphi_k = \nabla^2 \varphi_k = 0\), on \(\partial D\) and \(\tilde{\vartheta}_k = 0\) on \(\partial D \times I\).

Proof. We can without loss of generality assume that \(\int_D \psi_3^{h_n} \, dx = 0\), \(\forall n \in \mathbb{N}\) and every connected component \(D_i\) of \(D\), \(i = 1, \ldots, m\). By comparing with zero sequence one can additionally assume that 
\[
\|\text{sym } \nabla h_n \psi^{h_n}\|_{L^2(D)} \leq \eta_2 \|M\|_{L^2}^2 + 1, \quad \forall n \in \mathbb{N}.
\]
From Proposition 3.3 we have that there exist \((\tilde{\varphi}_n)_{n \in \mathbb{N}} \subset H^2(D)\) and \((\tilde{\psi}_n)_{n \in \mathbb{N}} \subset H^1(D \times I)\) such that 
\[
\text{sym } \nabla h_n \psi^{h_n} = -x_3 \ell(\nabla^2 \tilde{\varphi}_n) + \text{sym } \nabla h_n \tilde{\psi}_n + o^{h_n},
\]
where \(o^{h_n} \in L^2(D \times I, \mathbb{R}^{3 \times 3})\) and the following properties hold 
\[
\lim_{n \to \infty} \left(\|\tilde{\varphi}_n\|_{H^1(D)} + \|\tilde{\psi}_n\|_{L^2(D \times I)} + \|o^{h_n}\|_{L^2(D)}\right) = 0,
\]
\[
\limsup_{n \to \infty} \left(\|\tilde{\varphi}_n\|_{H^2(D)} + \|\tilde{\psi}_n\|_{L^2(D \times I)}\right) \leq C(D) \left(\eta_2 \|M\|_{L^2}^2 + 1\right).
\]
Now we use Proposition A.5 and Theorem A.6 to obtain the sequence \((\varphi_k)_{k \in \mathbb{N}} \subset H^2(D)\) and \((\tilde{\psi}_k)_{k \in \mathbb{N}}\) such that \((|\nabla^2 \varphi_k|^2)_{k \in \mathbb{N}}\) and \((|\nabla h_n(\tilde{\psi}_k)|^2)_{k \in \mathbb{N}}\) are equi-integrable and for \(A_k\) defined by 
\[
A_k := \{ \varphi_k \neq \varphi_n(\vartheta_{k,3}) \text{ or } \tilde{\psi}_n(\vartheta_{k,3}) \neq \tilde{\psi}_k \},
\]
we have \(|A_k| \to 0\) as \(k \to \infty\) and 
\[
\lim_{k \to \infty} \left(\|\varphi_k\|_{H^1(D)} + \|\tilde{\psi}_k\|_{L^2(D \times I)}\right) = 0.
\]
Define $\vartheta_k$ as in Proposition 3.3 by
\[
\vartheta_k := \tilde{\psi}_k + \begin{pmatrix} 0 \\ 0 \\ \tilde{\varphi}_k/h_n(k) \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_k \\ \partial_2 \varphi_k \\ 0 \end{pmatrix}.
\]
From the property [33], Remark [10] equi-integrability and the fact that $|A_k| \to 0$ as $k \to \infty$ we have the following
\[
K(M, D) = \lim_{n \to \infty} \int_{D \times I} Q^{h_n}(x, \iota(M) + \nabla_{h_n} \psi^{h_n}) \, dx
\]
\[
= \lim_{n \to \infty} \int_{D \times I} Q^{h_n}(x, \iota(M) - x_3 t(\nabla^2 \varphi^{h_n}) + \text{sym} \nabla_{h_n} \tilde{\psi}^{h_n}) \, dx
\]
\[
\geq \lim_{k \to \infty} \int_{A_k \times I} Q^{h_n(k)}(x, \iota(M) - x_3 t(\nabla^2 \varphi_k) + \text{sym} \nabla_{h_n(k)} \tilde{\psi}_k) \, dx
\]
\[
= \lim_{k \to \infty} \int_{D \times I} Q^{h_n(k)}(x, \iota(M) + \nabla_{h_n(k)} \vartheta_k) \, dx \geq K(M, D).
\]
The last inequality follows from the definition of $K(M, D)$ and the fact that $(\vartheta_{k,1}, \vartheta_{k,2}, \tilde{h}_n(k)^2 \vartheta_{k,3}) \to 0$ strongly in $L^2$. The last claim in (b) follows from the equi-integrability property and [40]. The last claim in (c) follows from simple truncation Lemma 3.6 below.

**Lemma 3.6.** Let $A \subset \omega$ be an open, bounded set. Let $\{\vartheta_n\}_{n \in \mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ be defined by
\[
\vartheta_n := \psi_n + \begin{pmatrix} 0 \\ 0 \\ \tilde{\varphi}_n/h_n \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n \\ \partial_2 \varphi_n \\ 0 \end{pmatrix},
\]
where $(\psi_n)_{n \in \mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ and $(\varphi_n)_{n \in \mathbb{N}} \subset H^2(A)$. Suppose that $(|\nabla^2 \varphi_n|^2)_{n \in \mathbb{N}}$ and $(|\nabla_{h_n} \psi_n|^2)_{n \in \mathbb{N}}$ are equi-integrable and
\[
\lim_{n \to \infty} \left( \|\varphi_n\|_{H^1(A)} + \|\psi_n\|_{L^2(A \times I)} \right) = 0.
\]
Then there exist sequences $(\tilde{\varphi}_n)_{n \in \mathbb{N}} \subset H^2(A), (\tilde{\psi}_n)_{n \in \mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ and a sequence of sets $(A_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $A_n \ll A_{n+1} \ll A$ and $\cup_{n \in \mathbb{N}} A_n = A$ and
(a) $\tilde{\varphi}_n = 0, \nabla \tilde{\varphi}_n = 0$ in a neighborhood of $\partial A$, $\tilde{\psi}_n = 0$ in a neighborhood of $\partial A \times I$.
(b) $\psi_n = \psi_n$ on $A_n \times I$, $\varphi_n = \varphi_n$ on $A_n$,
(c) $\|\tilde{\varphi}_n - \varphi_n\|_{H^2} \to 0$, $\|\psi_n - \psi_n\|_{H^1} \to 0$, $\|\nabla_{h_n} \tilde{\psi}_n - \nabla_{h_n} \psi_n\|_{L^2} \to 0$, as $n \to \infty$.
(d) for $\vartheta_n$ defined by
\[
\vartheta_n := \tilde{\psi}_n + \begin{pmatrix} 0 \\ 0 \\ \tilde{\varphi}_n/h_n \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n \\ \partial_2 \varphi_n \\ 0 \end{pmatrix},
\]
we have
\[
\lim_{n \to \infty} \left\| \text{sym} \nabla_{h_n} \vartheta_n - \text{sym} \nabla_{h_n} \tilde{\psi}_n \right\|_{L^2} = 0.
\]
**Proof.** By $\theta : [0, +\infty) \to [0, +\infty)$ denote the function:
\[
\theta(\varepsilon) = \sup_{n \in \mathbb{N}, S \subset A \atop \text{meas}(S) \leq \varepsilon} \left( \|\nabla^2 \varphi_n\|_{L^2(S)}^2 + \|\nabla_{h_n} \psi_n\|_{L^2(S \times I)}^2 \right).
\]
By the equi-integrability property we have $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. For fixed $k \in \mathbb{N}$ choose $A_k \ll A$ open set with Lipschitz boundary such that $\text{meas}(A \setminus A_k) \leq \frac{1}{k}$ and smooth cut-off.
function \( \eta_k \in C_0^\infty(A) \) such that \( 0 \leq \eta_k \leq 1 \) and \( \eta_k = 1 \) in a neighborhood of \( \bar{A}_k \). We can also assume that for every \( k \in \mathbb{N}, \ A_k \ll A_{k+1} \ll A \) and that \( \bigcup_{k \in \mathbb{N}} A_k = A \). Define \( \hat{\varphi}_{k,n} := \eta_k \varphi_n, \hat{\psi}_{k,n} := \eta_k \psi_n \). Define \( g : \mathbb{N} \times [0, +\infty) \to [0, +\infty) \) by

\[
g(k, n) = \| \hat{\varphi}_{k,n} - \varphi_n \|_{H^2} + \| \hat{\psi}_{k,n} - \psi_n \|_{H^1} + \| \nabla_{h_n} \hat{\psi}_{k,n} - \nabla_{h_n} \psi_n \|_{L^2}.
\]

Since we have for \( \alpha, \beta = 1, 2 \):

\[
\partial_{\alpha \beta} \hat{\varphi}_{k,n} = \partial_{\alpha \beta} \eta_k \varphi_n + \partial_{\alpha} \eta_k \partial_{\beta} \varphi_n + \eta_k \partial_{\alpha \beta} \varphi_n,
\]

\[
\partial_{\alpha} \hat{\psi}_{k,n} = \eta_k \partial_{\alpha} \psi_n + \partial_{\alpha} \eta_k \psi_n,
\]

\[
\partial_{3} \hat{\psi}_{k,n} = \eta_k \partial_{3} \psi_n,
\]

it is easy to conclude that there exists \( C > 0 \) such that for every \( k, n \in \mathbb{N} \) we have

\[
g(k, n) \leq C \left( \theta(\frac{1}{k}) + \| \eta_k \|_{C^2} \cdot (\| \varphi_n \|_{H^1} + \| \psi_n \|_{L^2}) \right).
\]

Since we also have, by the compactness, that \( \varphi_n \to 0 \), strongly in \( H^1 \) we conclude by the diagonalization argument that there exists a sequence \( k(n) \) monotonically increasing such that \( g(k(n), n) \to 0 \) as \( n \to \infty \). This proves (c). (d) follows directly from (c). \( \square \)

The following lemma is an easy consequence of Lemma 3.4 and Lemma 3.5. It is crucial for proving Proposition 2.10.

**Lemma 3.7.** Let \((h_n)_{n \in \mathbb{N}}\) be such that Assumption 2.3 is satisfied. The following properties are valid for every \( A, A_1, A_2 \subset \omega \) open sets and \( M, M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}) \):

(a) there exists \( K(M, A) \) such that we have

\[
K^+_1(h_n)(M, A) = K^-_1(h_n)(M, A) = K(M, A),
\]

(b) (localization) if \( A_1 \subset A_2 \) we have

\[
K(M \chi_{A_1 \times I}, A_2) = K(M, A_1),
\]

(c) (inner regularity)

\[
K(M, A) = \sup_{D \in P} K(M, D),
\]

(d) (boundedness)

\[
K(M, A) \leq \eta_2 \| M \|_{L^2(A \times I)}^2,
\]

(e) (monotonicity) if \( A_1 \subset A_2 \) we have

\[
K(M, A_1) \leq K(M, A_2),
\]

(f) (continuity) Let \((A_n)_{n \in \mathbb{N}}\) be the family of open subsets of \( \omega \) such that for each \( n \in \mathbb{N}, \ A_n \subset A_{n+1} \). Let \( A = \bigcup_{n=1}^{\infty} A_n \). Then \( \lim_{n \to \infty} K(M, A_n) = K(M, A) \).

(g) (additivity) if \( A_1 \cap A_2 = \emptyset \) then we have

\[
K(M, A_1 \cup A_2) = K(M, A_1) + K(M, A_2),
\]

(h) \[
|K(M_1, A) - K(M_2, A)| \leq C \| M_1 - M_2 \|_{L^2} (\| M_1 \|_{L^2} + \| M_2 \|_{L^2}),
\]

where \( C \) depends on \( \alpha, \beta \).

(i) (homogeneity)

\[
K(t M, A) = t^2 K(M, A), \forall t \in \mathbb{R}.
\]

(j) (paralelogram inequality)

\[
K(M_1 + M_2, A) + K(M_1 - M_2, A) \leq 2K(M_1, A) + 2K(M_2, A),
\]

for every \( M_1, M_2, A \subset \mathbb{R}^{2 \times 2}_{\text{sym}} \).
(k) (coerciveness) \[ K(M, A) \geq \eta_1 \|M\|_{L^2(A \times I)}^2. \]

(l) (subadditivity) if \( A \subset A_1 \cup A_2 \) we have \[ K(M, A) \leq K(M, A_1) + K(M, A_2). \]

Proof. Using Lemma 3.5, it is easy to see that for \( A \subset \omega \) open and an arbitrary \( D \in \mathcal{D}, D \subset A \) we have

\[ K^+(M_{\chi_D \times I}, A) = K^-(M_{\chi_D \times I}, A) = K(M, D), \]

Namely, the inequality \( K^+(M_{\chi_D \times I}, A) \geq K(M, D) \), follows immediately from the definition in Remark 3. To prove the inequality \( K^+(M_{\chi_D \times I}, A) \leq K(M, D) \) it is enough to prove that for every \( r > 0 \) we have \( \limsup_{n \to \infty} K_n(M, D, A, B(r)) \leq K(M, D) \), where

\[ K_n(M, D, A, B(r)) = \inf_{\psi \in H^1(A \times I, \mathbb{R}^3) \atop \|\psi_{1,2,3} h_n\psi\|_{L^2} \leq r} \int_{A \times I} Q^{h_n} (x, \iota(M_{\chi_D \times I}) + \nabla h_n \psi) \, dx. \]

To prove this take a subsequence, still denoted by \((h_n(k))_{k \in \mathbb{N}}\), where \( \limsup \) is accomplished. Then we take, using Lemma 3.5 a further subsequence, still denoted by \((h_n(k))_{k \in \mathbb{N}}\), and \((\psi_k)_{k \in \mathbb{N}} \subset H^1(D \times I, \mathbb{R}^3)\) such that \( (\psi_{k,1}, \psi_{k,2}, h_n(k)\psi_{k,3}) \to 0 \), strongly in \( L^2 \), for each \( k \in \mathbb{N}, \chi_k = 0 \) on \( \partial D \times I \) and

\[ K(M, D) = \lim_{k \to \infty} \int_{D \times I} Q^{h_n(k)} (x, \iota(M_{\chi_D \times I}) + \nabla h_n(k) \psi_k) \, dx. \]

By extending \( \chi_k = 0 \) on \((A \setminus D) \times I\) we obtain

\[ K(M, D) = \lim_{k \to \infty} \int_{A \times I} Q^{h_n(k)} (x, \iota(M_{\chi_D \times I}) + \nabla h_n(k) \psi_k) \, dx \geq \lim_{k \to \infty} K_n(k)(M, D, A, B(r)). \]

By the arbitrariness of \( r \) we have the claim. (a) and (b) follow from (12) and Lemma 3.4 by an approximation argument i.e. by exhausting \( A \) with the sets in \( \mathcal{D} \). It is easy to notice from the definition in Remark 3 that \( K(M, D) \leq K(M, A) \), for \( D \in \mathcal{D}, D \subset A \). (c) then easily follows from (b) and Lemma 3.4. From the definition in Remark 3 by taking the null sequence, it is easy to see that for every \( D \in \mathcal{D} \) we have \( K(M, D) \leq \eta_2 \|M\|_{L^2(D)}^2 \).

(d) now follows from (c). (e) easily follows from (c). (f) is again the direct consequence of (b) and Lemma 3.4. To prove (g) first choose \( D_1, D_2 \in \mathcal{D}, D_1 \ll A_1 \) and \( D_2 \ll A_2 \). We have that \( D_1 \cap D_2 = \emptyset \). From the definition in Remark 3 it is easy to see that

\[ K(M, D_1 \cup D_2) = K(M, D_1) + K(M, D_2). \]

(g) now follows from (f). To prove (h) notice that from Lemma 3.4 we can conclude that there exists \( C > 0 \) dependent only on \( \alpha, \beta \) such that for each \( D \in \mathcal{D} \) we have

\[ |K(M_1, D) - K(M_2, D)| \leq C \|M_1 - M_2\|_{L^2} (\|M_1\|_{L^2} + \|M_2\|_{L^2}), \]

\[ \forall M_1, M_2 \in L^2(\omega, \mathbb{R}^{2 \times 2}_{\text{sym}}). \]

(h) follows from (f) and (43). To prove (i) we can take \( D \in \mathcal{D} \) and \( M \in L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}) \). Define by

\[ K_n(M, D, B(r)) = \min_{\psi \in H^1(D \times I, \mathbb{R}^3) \atop \|\psi_{1,2,3} h_n\psi\|_{L^2} \leq r} \int_{D \times I} Q^{h_n} (x, \iota(M) + \nabla h_n \psi) \, dx. \]
The minimum in the above expression exists by the direct methods of the calculus of variation. Notice that
\[ K(M, D) = \lim_{n \to 0} \lim_{n \to \infty} K_n(M, D, B(r)), \]
where limit in \( n \) can be taken on any converging subsequence (dependent on \( r \)). Since every \( Q^{h_n} \) is quadratic we have
\[ t^2 K_n(M, D, B(r)) = K_n(tM, D, B(|t|r)). \]
From this identity it follows \( K(tM, D) = t^2 K(M, D) \). By approximation we obtain (i).
To prove (j) take \( M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2 \times 2}), D \in \mathcal{D} \) and for \( \alpha = 1, 2, \psi^{\alpha,r,n} \in H^1(D \times I, \mathbb{R}^3) \) such that
\[ K_n(M_\alpha, D, B(r)) = \int_{D \times I} Q^{h_n}(x, \iota(M_\alpha) + \nabla h_n \psi^{\alpha,r,n}) \, dx. \]
and \( \| (\psi_1^{\alpha,r,n}, \psi_2^{\alpha,r,n}, h_n \psi_3^{\alpha,r,n}) \|_{L^2} \leq r \). Notice that:
\[ K_n(M_1 + M_2, D, B(2r)) + K_n(M_1 - M_2, D, B(2r)) \]
\[ \leq \int_{D \times I} Q^{h_n}(x, \iota(M_1 + M_2) + \nabla h_n \psi_1^{1,r,n} + \nabla h_n \psi_2^{2,r,n}) \, dx \]
\[ + \int_{D \times I} Q^{h_n}(x, \iota(M_1 - M_2) + \nabla h_n \psi_1^{1,r,n} - \nabla h_n \psi_2^{2,r,n}) \, dx \]
\[ = 2 \int_{D \times I} Q^{h_n}(x, \iota(M_1) + \nabla h_n \psi_1^{1,r,n}) \, dx + 2 \int_{D \times I} Q^{h_n}(x, \iota(M_2) + \nabla h_n \psi_2^{2,r,n}) \, dx \]
\[ = 2K_n(M_1, D, B(r)) + 2K_n(M_2, D, B(r)), \]
where we have used (e) of Proposition [A.7]. By letting \( n \to \infty \) and then \( r \to 0 \) we obtain that
\[ K(M_1 + M_2, D) + K(M_1 - M_2, D) \leq 2K(M_1, D) + 2K(M_2, D). \]
(j) follows by density and (f). To prove (k) take \( M \in C^1(\Omega, \mathbb{R}^{2 \times 2}) \), such that \( M = 0 \) in a neighborhood of \( \partial \Omega \), \( D \in \mathcal{D} \) and \( (\psi_n)_{n \in \mathbb{N}} \subset H^1(D \times I, \mathbb{R}^3) \) such that \( (\psi_{n,1}, \psi_{n,2}, h_n \psi_{n,3}) \to 0 \), strongly in \( L^2 \) and such that
\[ K(M, D) = \lim_{n \to \infty} \int_{D \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi_n) \, dx. \]
From \((Q1)\) we conclude
\[ \int_{D \times I} Q^{h_n}(x, \iota(M) + \nabla h_n \psi_n) \, dx \geq \eta \int_{D \times I} |M + \text{sym} \nabla'(\psi_{n,1} \cdot \psi_{n,2})|^2 \, dx \]
\[ \geq \eta \| M \|_{L^2}^2 - \eta \int_{D \times I} \text{div} M \cdot (\psi_{n,1} \cdot \psi_{n,2}) \, dx. \]
By letting \( n \to \infty \) and using the fact that \( (\chi_{n,1}, \chi_{n,2}) \to 0 \) strongly in \( L^2 \) we have that \( K(M, D) \geq \eta_1 \| M \|_{L^2}^2 \). (k) follows from (f) and (h). To prove (l) we proceed as follows: by using (c) and (e) it is enough to prove that for arbitrary \( D, D_1, D_2 \in \mathcal{D} \) such that \( D \subset D_1 \cup D_2 \) we have
\[ K(M, D) \leq K(M, D_1) + K(M, D_2). \]
Take \( D_2' \in \mathcal{D} \) such that \( D_2' \ll D_2 \setminus D_1 \). From (f) we have
\[ K(M, D_1 \cup D_2') = K(M, D_1) + K(M, D_2'). \]
From (g) we have for some $C > 0$

\[ K(M, D) = K(M \chi_{D \times I}, \omega) \]
\[ \leq K(M \chi_{(D_1 \cup D'_2) \times I}, \omega) + C\|M\|_{L^2((D_2 \setminus (D_1 \cup D'_2) \times I))} \|M\|_{L^2} \]
\[ = K(M, D_1) + K(M, D'_2) + C\|M\|_{L^2((D_2 \setminus (D_1 \cup D'_2) \times I))} \|M\|_{L^2} \]
\[ = K(M, D_1) + K(M, D_2) + C\|M\|_{L^2((D_2 \setminus (D_1 \cup D'_2) \times I))} \|M\|_{L^2}. \]

The claim follows by the arbitrariness of $D'_2$.

\[ \square \]

At the end of this section we improve Lemma 3.5 for arbitrary $A \subset \Omega$ open.

**Lemma 3.8.** Let Assumption 2.8 be satisfied for a sequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero. Take $M \in L^2(\Omega, \mathbb{R}^{2 \times 2})$ and $A \subset \Omega$ open. Then there exists a subsequence $(h_{n(k)})_{k \in \mathbb{N}}$ and $(\vartheta_k)_{k \in \mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ such that

(a) $(\vartheta_{k,1}, \vartheta_{k,2}, h_{n(k)} \vartheta_{k,3}) \to 0$ strongly in $L^2$,

(b) $(\Delta \nabla_{h_{n(k)}} \vartheta_{k})_{k \in \mathbb{N}}$ is equi-integrable,

where $(\Delta \nabla_{h_{n(k)}} \vartheta_{k})_{k \in \mathbb{N}}$ are equi-integrable and $\vartheta_k \to 0$ strongly in $H^1$ and $\tilde{\vartheta}_k \to 0$ strongly in $L^2$. Also the following holds

\[ \limsup_{k \to \infty} (\|\vartheta_k\|_{H^2(A)} + \|\nabla_{h_{n(k)}} \tilde{\vartheta}_k\|_{L^2(A)}) \leq C (\|M\|_{L^2}^2 + 1), \]

where $C$ is independent of the domain $A$ and for each $k \in \mathbb{N}$ we have $\vartheta_k = 0$ in a neighborhood of $\partial A \times I$, i.e., $\vartheta_k = \nabla^2 \varphi_k = 0$, in a neighborhood of $\partial A$ and $\tilde{\vartheta}_k = 0$ in a neighborhood of $\partial A \times I$.

(c)

\[ K(M, A) = \lim_{k \to \infty} \int_{A \times I} Q^{h_{n(k)}}(x, \iota(M) + \nabla_{h_{n(k)}} \vartheta_k) \, dx. \]

**Proof.** Take $r > 0$ such that $B(r) \supsetneq \omega$. Extend $Q^h$ on $(B(r) \setminus \omega) \times I$ by e.g. $Q^h(x, G) = \eta_2 |\nabla G|^2$, for all $x \in (B(r) \setminus \omega) \times I$. Apply Lemma 3.5 to $M = M \chi_A$ and $D = B(r)$ to obtain the sequences $((\tilde{\vartheta}_k)_{k \in \mathbb{N}} \subset H^1(B(r) \times I, \mathbb{R}^3)), (\tilde{\varphi}_k)_{k \in \mathbb{N}} \subset H^2(B(r)), (\tilde{\psi}_k)_{k \in \mathbb{N}} \subset H^1(B(r) \times I, \mathbb{R}^3)$ that satisfy (a), (b), (d) of Lemma 3.5. In the same way as in Lemma 3.6 for each $\varepsilon > 0$ we choose $A_\varepsilon \ll A$ with Lipschitz boundary such that $\text{meas}(A \setminus A_\varepsilon) < \varepsilon$ and a cut off function $\eta_\varepsilon \in C_0^\infty(A)$ which is 1 on $A_\varepsilon$. Again by using the diagonal procedure, we obtain a sequence $(\varphi_k)_{k \in \mathbb{N}} \subset H^2(A)$ and $(\psi_k)_{k \in \mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ such that for each $k \in \mathbb{N}$, $\varphi_k = \nabla^2 \varphi_k = 0$ in a neighborhood of $B(r) \setminus A$, i.e., $\varphi_k = 0$ in a neighborhood of $(B(r) \setminus A) \times I$ and $\psi_k = 0$ in a neighborhood of $(B(r) \setminus A) \times I$ and

\[ \lim_{k \to \infty} (\|\tilde{\varphi}_k - \varphi_k\|_{H^2(A)} + \|\tilde{\psi}_k - \varphi_k\|_{L^2(A \times I}) + \|\nabla_{h_{n(k)}} \tilde{\psi}_k - \nabla_{h_{n(k)}} \varphi_k\|_{L^2(A \times I)}) = 0. \]

Define again

\[ \vartheta_k := \tilde{\psi}_k + \begin{pmatrix} 0 \\ 0 \\ \varphi_k \\ h_{n(k)} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_k \\ \partial_2 \varphi_k \\ 0 \end{pmatrix}. \]
It is easy to see from (b) of Lemma 3.7 that
\[
K(M, A) = \int_{B(r) \times I} Q^h_n(x, t(M\chi_{A \times I}) + \nabla h_n(x) \tilde{\vartheta}_k) \, dx
\]
\[\geq \lim_{k \to \infty} \int_{A \times I} Q^h_n(x, t(M) + \nabla h_n(x) \tilde{\vartheta}_k) \, dx \geq K(M, A).
\]
From this we have the claim. \(\square\)

3.3. Proof of Proposition 2.10 and Corollary 2.11

Proof of Proposition 2.10 \[2.10\] The proof follows the standard steps in \(\Gamma\)-convergence theory.

Notice that for \(M \in \mathcal{S}_{vK}(\Omega)\)
\[
\|M\|_{L^2(\Omega)}^2 = \|M_1\|_{L^2(\omega)}^2 + \frac{1}{12} \|M_2\|_{L^2(\omega)}^2.
\]

Step 1. Existence of \(Q\).
From Theorem A.11 as well as the properties (d), (e), (g), (f) and (l) from Lemma 3.7 we conclude from Radon-Nykodim theorem that for an arbitrary \(M \in \mathcal{S}_{vK}(\Omega)\) there exists \(Q_M \in L^1(\omega)\), a positive function, such that
\[
K(M, A) = \int_{A} Q_M(x) \, dx', \forall A \subset \omega \text{ open}.
\]

Take a countable dense subset \(\mathcal{M}\) of \(\mathbb{R}^{2 \times 2}_{\text{sym}}\) and define
\[
E := \{x' \in \omega : x' \text{ is a Lebesgue point for } Q_{M_1 + x_3 M_2} \text{ for every } M_1, M_2 \in \mathcal{M}\}.
\]

Notice that \(\text{meas}(\omega \setminus E) = 0\). Define also
\[
Q(x', M_1, M_2) = Q_{M_1 + x_3 M_2} (x') = \lim_{r \to 0} \frac{1}{|B(x', r)|} K\left(M_1 + x_3 M_2, B(x', r)\right),
\]
for \(M_1, M_2 \in \mathcal{M}\) and \(x' \in E\).

Notice that from the property (h) in Lemma 3.7 we have
\[
|Q(x', M_1, M_2) - Q(x', M'_1, M'_2)| \leq C\left(|M_1 - M'_1| + |M_2 - M'_2|\right)\left(|M_1 + M'_1| + |M_2 + M'_2|\right),
\]
for all \(M_1, M'_1, M_2, M'_2 \in \mathcal{M}, x' \in E\).

Thus, we can extend \(Q(\cdot, \cdot, \cdot)\) by continuity on \(E \times \mathbb{R}^{2 \times 2}_{\text{sym}} \times \mathbb{R}^{2 \times 2}_{\text{sym}}\). Extend it by zero on \(\omega \times \mathbb{R}^{2 \times 2}_{\text{sym}} \times \mathbb{R}^{2 \times 2}_{\text{sym}}\). Notice that such defined \(Q\) satisfies
\[
|Q(x', M_1, M_2) - Q(x', M'_1, M'_2)| \leq C\left(|M_1 - M'_1| + |M_2 - M'_2|\right)\left(|M_1 + M'_1| + |M_2 + M'_2|\right),
\]
for all \(M_1, M'_1, M_2, M'_2 \in \mathbb{R}^{2 \times 2}_{\text{sym}}, x' \in \omega\).

(49) Also, from the property (h) in Lemma 3.7 and (48) we conclude that
\[
Q(x', M_1, M_2) = \lim_{r \to 0} \frac{1}{|B(x', r)|} K\left(M_1 + x_3 M_2, B(x', r)\right), \forall M_1, M_2 \in \mathbb{R}^{2 \times 2}_{\text{sym}} \text{ and } x' \in E.
\]

By approximating \(M \in \mathcal{S}_{vK}(\Omega)\) by piecewise constant maps with values in the set \(\{M_1 + x_3 M_2 : M_1, M_2 \in \mathcal{M}\}\) we conclude from (b), (g), (h) of Lemma 3.7 as well as the properties (48) and (49) that
\[
K(M, \omega) = \int_{\omega} Q(x', M_1(x'), M_2(x')) \, dx', \forall M \in \mathcal{S}_{vK}(\omega).
\]
By using (b) of Lemma 3.7 as well as the fact that $Q(x',0,0) = 0 \forall x' \in \omega$ we conclude (46).

**Step 2. Quadraticity and coercivity of $Q$.
** To prove that $Q$ is quadratic form we use (50), Proposition A.7 and (i), (j) of Lemma 3.7. To prove coercivity we use (45), (50) and property (k) of Lemma 3.7.

**Proof of Corollary 2.11.** By Remark 5 and Lemma 2.9 it is enough to see that every sequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero has a subsequence $(h_{n(k)})_{k \in \mathbb{N}}$ such that

$$K(h_{n(k)})(M,D) =: K(M,D), \forall n \in \mathbb{N}, M \in S_{\kappa K}(\omega), D \in D,$$

where $K(M,D)$ is independent of the sequence. This follows from Lemma 2.7, Remark 5 and Proposition 2.10, i.e., (46) and (47).

**3.4. Proof of Theorem 2.12.** Denote by $e^h(y)$

$$e^h(y) = \frac{1}{R^4} \int_{\Omega} \text{dist}^2(\nabla_h y, SO(3)).$$

The proof of the following proposition is given in [NV13, Proposition 3.1]. It characterizes the deformations which have the order of the energy $h^4$.

**Proposition 3.9.** Let $y \in H^1(\Omega, \mathbb{R}^3)$ and $h > 0$. There exist $(\bar{R}, u, v) \in SO(3) \times H^1(\omega, \mathbb{R}^2) \times H^2_{bc}(\omega)$ and correctors $w \in H^1(\omega), \psi \in H^1(\Omega, \mathbb{R}^3)$ with

$$\int_{\omega} w = 0, \quad \int_{I} \psi(\hat{x}, x_3) \, dx_3 = 0 \quad \text{for almost every } x' \in \omega,$

such that

$$\bar{R}^t \left( y - \int_{\Omega} y \, dx \right) = \left( \begin{array}{c} x' \\ h x_3 \end{array} \right) + \left( \begin{array}{c} h^2 u \\ h(v + h w) \end{array} \right) - h^2 x_3 \left( \begin{array}{c} \hat{\nabla} \psi \\ 0 \end{array} \right) + h^2 \psi$$

and

$$\|u\|_{H^1(\omega)}^2 + \|v\|_{H^1(\omega)}^2 + \|w\|_{H^2(\omega)}^2 + \frac{1}{R^2} \|\psi\|_{L^2(\Omega)}^2 \leq C(\omega)(e^h(y) + e^h(y)^2).$$

In addition, for all $D \ll \omega$ compactly contained in $\omega$ we have

$$\|\nabla^2 v\|_{L^2(D)}^2 + \|\nabla h \psi\|_{L^2(D \times I)}^2 \leq C(D)(e^h(y) + e^h(y)^2).$$

If the boundary of $\omega$ is of class $C^{1,1}$, then $(u, v) \in A(\omega)$ and (55) holds for $D$ replaced by $\omega$.

The following lemma is an easy consequence of Taylor expansion. It is the essential part of lower bound theorem.

**Lemma 3.10.** Let $G^h, K^h \in L^2(\Omega, \mathbb{R}^{3 \times 3})$ be such that

$$K^h \text{ is skew-symmetric and}$$

$$\limsup_{h \to 0} \left( \|G^h\|_{L^2} + \|K^h\|_{L^1} \right) < \infty,$$

Consider

$$E^h := \sqrt{(Id + hK^h + h^2G^h)^t(Id + hK^h + h^2G^h) - Id}/h^2.$$
Then there exists a sequence \((\chi^h)_{h>0}\) such that, \(\chi^h : \Omega \to \{0, 1\}\), \(\chi^h \to 1\) boundedly in measure and
\[
\lim_{h \to 0} \left\| \chi^h \left( E^h - \left( \text{sym } G^h - \frac{1}{2} (K^h)^2 \right) \right) \right\|_{L^2} = 0.
\]

**Proof.** Notice that the following claim is the direct consequence of Taylor expansion: There exists \(\delta > 0\) and a monotone increasing function \(\eta : (0, \delta) \to (0, \infty)\) such that \(\lim_{r \to 0} \eta(r) = 0\) and
\[
(58) \quad \left|\sqrt{(Id + A)^t (Id + A)} - (Id + \text{sym } A + \frac{1}{2} A^t A)\right| \leq \eta(|A|) \left(\text{sym } A + \frac{1}{2} A^t A\right)
\]
\(\forall A \in \mathbb{R}^{3\times 3}, |A| < \delta\).

Now we use the truncation argument. Namely, let \(\chi\) be the characteristic function of the set \(S\), where
\[
S^h = \{x \in \Omega : |K^h| \leq \frac{1}{\sqrt{h}}, \quad |G^h| \leq \frac{1}{h}\}.
\]
The claim follows after putting \(A = hK^h + h^2 G^h\) into the expression (58), dividing by \(h^2\) and letting \(h \to 0\). \(\square\)

We state one simple linearization lemma, whose proof is analogous to [NV13, Lemma 3.1]. We shall skip it here.

**Lemma 3.11** (linearization). Let \(\{\tilde{E}^h\}_{h>0} \subset L^2(\Omega, \mathbb{R}^{3\times 3})\) satisfy
\[
(59) \quad \limsup_{h \to 0} \left\| \tilde{E}^h \right\|_{L^2} < \infty \quad \text{and} \quad \lim_{h \to 0} h^2 \|\tilde{E}^h\|_{L^\infty} = 0.
\]
Then
\[
\lim_{h \to 0} \left| \frac{1}{h^3} \int_{\Omega} W^h(x, Id + h^2 \tilde{E}^h(x)) \, dx - \int_{\Omega} Q^h(x, \tilde{E}^h(x)) \, dx \right| = 0.
\]

**Proof of Theorem 2.12.** The key fact is to obtain from Proposition 3.9 the representation of the strain in the form
\[
(60) \quad E^{hn} = \text{sym } \nabla u - \frac{1}{2} \nabla v \otimes \nabla v - x_3 \nabla^2 v + \nabla r^2 \varphi^{hn} + \nabla h_n \hat{\psi}^{hn} + o^{hn},
\]
on a large set that vanishes as \(n \to \infty\). Here
\[
\lim_{n \to \infty} \|o^{hn}\|_{L^2} = \lim_{n \to \infty} \|\hat{\psi}^{hn}\|_{L^2} = \lim_{n \to \infty} \|\varphi^{hn}\|_{H^1} = 0,
\]
\[
\limsup_{n \to \infty} \|\varphi^{hn}\|_{H^2} < \infty, \quad \limsup_{n \to \infty} \|\nabla h_n \hat{\psi}^{hn}\|_{L^2} < \infty.
\]
This is established with the relations (62), (65), (66) and (67). To make the lower bound we have to modify the sequences \((|\nabla^2 \varphi^{hn}|^2)_{n \in \mathbb{N}}\) and \((|\nabla h_n \hat{\psi}^{hn}|^2)_{n \in \mathbb{N}}\) by equi-integrable ones.

For the proof we can take \(R = Id\). First we assume that \(\omega\) is of class \(C^2\). Without loss of generality we assume that
\[
(61) \quad \liminf_{n \to \infty} I^{hn}(y^{hn}) = \limsup_{n \to \infty} I^{hn}(y^{hn}) < \infty.
\]
Due to the non-degeneracy of \(W^{hn}\) (see (W2)) we have \(\limsup_{n \to \infty} e^{hn}(y^{hn}) < \infty\). Hence, Proposition 3.9 is applicable, and we easily deduce the part (i), by taking in the expression (53) the integral over the interval \(I\). From the estimate (54) and (55) we conclude that
\[
v^{hn} \to v\quad \text{weakly in } H^2, \quad u^{hn} \to u\quad \text{weakly in } H^1.
\]
Notice that from the expression (53) we have that
\[ \nabla_{h_n} y_{h_n} = \text{Id} + h_n K_{h_n} + h_n^2 G_{h_n}, \]
where
\[ K_{h_n} = \begin{pmatrix} 0 & 0 & -\partial_1 v_{h_n} \\ 0 & 0 & -\partial_2 v_{h_n} \\ \partial_1 v_{h_n} & \partial_2 v_{h_n} & 0 \end{pmatrix}, \]
\[ G_{h_n} = \iota (-x_3 \nabla^2 v_{h_n} + \nabla' u_{h_n}) + \nabla_{h_n} \psi_{h_n} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 w_{h_n} & \partial_2 w_{h_n} & 0 \end{pmatrix}. \]

Notice that \( K_{h_n}, G_{h_n} \) satisfy the hypothesis of Lemma 3.10. Define also
\[ E_{h_n} = \sqrt{(\nabla_{h_n} y_{h_n})^t \nabla_{h_n} y_{h_n} - \text{Id}}. \]

From the inequality valid for any \( F \in \mathbb{R}^{3 \times 3}, |\sqrt{F^t F} - \text{Id}| \leq \text{dist}(F, \text{SO}(3)), \) where the equality holds for \( F \in \mathbb{R}^{3 \times 3} \) such that \( \det F > 0, \) we conclude that \( \limsup_{n \to \infty} \|E_{h_n}\|_{L^2} < \infty. \) We truncate the peaks of \( E_{h_n} \) and the set of points where \( \det \nabla_{h_n} y_{h_n} \) is negative. Therefore, consider the good set \( C_{h_n} := \{ x \in \Omega : |E_{h_n}(x)| \leq h_n^{-1}, \det \nabla_{h_n} y_{h_n}(x) > 0 \} \) and let \( \chi_{1 n} \) denote the indicator function associated with \( C_{h_n}. \) It is easy to see that \( \chi_{1 n} \to 1 \) boundedly in measure. By applying Lemma 3.10, we know that there exists a sequence \( (\chi_{2 n})_{n \in \mathbb{N}} \) such that for all \( n, \chi_{2 n} : \Omega \to \{0, 1\}, \chi_{2 n} \to 1 \) boundedly in measure
\[ \lim_{n \to \infty} \left\| \chi_{2 n} \left( E_{h_n} - \left( \text{sym} G_{h_n} - \frac{1}{2} (K_{h_n})^2 \right) \right) \right\|_{L^2} = 0. \]

In the same way as in Proposition 3.3, we take \( (\tilde{w}_{h_n})_{n \in \mathbb{N}} \) such that
\[ \lim_{n \to \infty} \|w_{h_n} - \tilde{w}_{h_n}\|_{L^2} = 0, \limsup_{n \to \infty} \|\tilde{w}_{h_n}\|_{H^1} \leq C(\omega) \limsup_{n \to \infty} \|w_{h_n}\|_{H^1}, \lim_{n \to \infty} h_n\|\tilde{w}_{h_n}\|_{H^2} = 0. \]

Also we take the sequence \( (\tilde{v}_{h_n})_{n \in \mathbb{N}} \subset C^2(\omega), \) such that
\[ \|\tilde{v}_{h_n} - v\|_{H^2} \to 0, \quad h_n\|\tilde{v}_{h_n}\|_{C^2} \to 0. \]
This can be done by taking a smooth sequence converging to \( v \) and reparametrizing it. Notice that
\[ \text{sym} G - \frac{1}{2} (K_{h_n})^2 = \iota (M_1 + x_3 M_2) - x_3 \iota \left( \nabla^2 (v_{h_n} - v) \right) + \text{sym} \nabla_{h_n} \tilde{v}_{h_n} + o_{h_n}, \]
where

\[ M_1 = \text{sym} \nabla' u - \frac{1}{2} \nabla' v \otimes \nabla' v, \]

\[ M_2 = -\nabla'^2 v, \]

\[ \tilde{\psi}^h_n = \psi^h_n + \left( \frac{u^h_n - u_1}{u^h_n - u_2} \right) \right) + hx_3 \left( \begin{array}{c} \partial_1 \tilde{w}^h_n \\ \partial_2 \tilde{w}^h_n \\ -\frac{1}{2} \left( |\partial_1 \tilde{v}^h_n|^2 + |\partial_2 \tilde{v}^h_n|^2 \right) \end{array} \right); \]

\[ \phi^h_n = -\frac{1}{2} t \left( \nabla' v^h_n \otimes \nabla' v^h_n - \nabla' v \otimes \nabla' v \right) - h_n x_3 t (\nabla'^2 w^h_n) \]

\[ + \frac{1}{2} \text{sym} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & h_n x_3 \nabla' (|\partial_1 \tilde{v}^h_n|^2 + |\partial_2 \tilde{v}^h_n|^2) & 0 \\ 0 & 0 & h_n x_3 \nabla' (|\partial_1 \tilde{v}^h_n|^2 + |\partial_2 \tilde{v}^h_n|^2 - |\partial_1 v^h_n|^2 - |\partial_2 v^h_n|^2) \end{array} \right). \]

Notice that from (54), (55) as well as (63) and (64) and Sobolev embedding we conclude

\[ \lim_{n \to \infty} \|\psi^h_n\|_{L^2} = \lim_{n \to \infty} \|\tilde{\psi}^h_n\|_{L^2} = \lim_{n \to \infty} \|v^h_n - v\|_{H^1} = 0, \]

\[ \limsup_{n \to \infty} \|\tilde{\psi}^h_n - v^h_n\|_{H^2} < \infty, \quad \limsup_{n \to \infty} \|\nabla h_n \tilde{\psi}^h_n\|_{L^2} < \infty. \]

By using Proposition A.5 and Theorem A.6 we find a subsequence \((h_n(k))_{k \in \mathbb{N}}\) and sequences

\[ (\varphi_k)_{k \in \mathbb{N}} \subset H^2(\omega), \quad (\tilde{\psi}_k)_{k \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3), \quad (\chi_{3,k})_{k \in \mathbb{N}} \] such that

\[ \lim_{k \to \infty} \|\varphi_k\|_{H^1} = \lim_{k \to \infty} \|\tilde{\psi}_k\|_{L^2} = 0, \]

\[ \left( |\nabla^2 \varphi_k|^2 \right)_{k \in \mathbb{N}}, \left( |\nabla h_n(k) \tilde{\psi}_k|^2 \right)_{k \in \mathbb{N}} \] are equi-integrable,

\[ \chi_{3,k} : \Omega \to \{0, 1\}, \forall k, \chi_{3,k} \to 1 \text{ boundedly in measure}, \]

\[ \{x = (x', x_3) \in \Omega : \varphi_k(x') \neq (v^h_n(k) - v)(x') \} \quad \text{or} \quad \tilde{\psi}_k(x) \neq \tilde{\psi}^h_n(k)(x) \} = \{\chi_{3,k} = 0\}. \]

Define

\[ \vartheta_k = \tilde{\psi}_k + \left( \begin{array}{c} 0 \\ 0 \\ x_3 \end{array} \right) \left( \begin{array}{c} \partial_1 \varphi_k \\ \partial_2 \varphi_k \\ 0 \end{array} \right), \]

and notice that

\[ \text{sym} \nabla h_n(k) \vartheta_k = -x_3 t (\nabla'^2 \varphi_k + \text{sym} \nabla h_n(k) \tilde{\psi}_k). \]

From this and (69) and (71) we conclude that the family \((\text{sym} \nabla h_n(k) \vartheta_k)_{k \in \mathbb{N}}\) is equi-integrable and

\[ \{x \in \Omega : \text{sym} \nabla h_n(k) \vartheta_k \neq -x_3 t (\nabla'^2 (v^h_n(k) - v) + \nabla h_n(k) \tilde{\psi}^h_n(k)) \} = \{\chi_{3,k} = 0\}, \]

up to a set of measure zero (see Remark 10). From (68) we conclude that \((\vartheta_{k,1}, \vartheta_{k,2}, h_n(k) \vartheta_{k,3}) \to 0 \) strongly in \(L^2\). Now we are ready to prove the lower bound. The key idea is that the replacement by equi-integrable family enables us to establish the lower bound on the whole set. Denote by

\[ \chi_k = \chi_{h_n(k)} h_n(k) \chi_{3,k}, \quad \tilde{E}_k = \chi_k E^h_n(k). \]

By appealing to the polar factorization for matrices with non-negative determinant, there exists a matrix field \(R^h_n : C^{h_n} \to \text{SO}(3)\) such that

\[ \forall x \in C^{h_n} : \nabla h_n y^{h_n}(x) = R^h_n(x) \sqrt{(\nabla h_n y^{h_n}(x))^t \nabla h_n y^{h_n}(x)}. \]
Hence, by frame-indifference (see (W1)), non-negativity (see (W2)) and assumption (W3) we have
\[ W^{h_n(k)}(x, \nabla h_n(k)y^{h_n(k)}(x)) \geq \chi_k(x)W(x, \nabla h \hat{y}^h(x)) = W^{h_n(k)}(x, \text{Id} + h_n^{2(k)}\tilde{E}_k(x)). \]
Thus,
\[ I_n^{h_n(k)}(y^{h_n(k)}) = \frac{1}{h_n^{4(k)}} \int_\Omega W^{h_n(k)}(x, \nabla h_n(k)y^{h_n(k)}(x)) \, dx \geq \frac{1}{h_n^{4(k)}} \int_\Omega W^{h_n(k)}(x, \text{Id} + h_n^{2(k)}\tilde{E}_k(x)) \, dx. \]
Due to the truncation we have \( \lim_{k \to \infty} h_n(k)^2\|\hat{E}_k\|_{L^\infty} = 0 \). Hence, using [3], Lemma [3.11] and the equi-integrability of \( (\|\text{sym} \nabla h_n(k)\vartheta_k^2\|_{k \in \mathbb{N}}) \) with (Q1) as well as \([61],[62],[65],[66],[72]\) and Proposition 2.10 we get
\[ \liminf_{n \to \infty} I_n^{h_n(y^{h_n})} = \liminf_{k \to \infty} I_n^{h_n(k)}(y^{h_n(k)}) \geq \liminf_{k \to \infty} \int_\Omega Q^{h_n(k)}(x, \hat{E}_k(x)) \, dx \]
\[ \geq \int_\Omega Q(x', M_1, M_2) \, dx' = I^0(u, v). \]
To deal with arbitrary \( \omega \) Lipschitz one firstly takes \( D \ll \omega \) of class \( C^2 \) and conclude that \( u \in H^1(\omega, \mathbb{R}^2), v \in H^1(\omega) \cap H^2(D) \). In the same way as above we conclude
\[ \lim_{n \to \infty} I_n^{h_n(y^{h_n})} \geq \int_D Q(x', M_1, M_2) \, dx' \geq \frac{\eta_1}{12} \|\nabla^2 u\|_{L^2(D)}, \] where we have used (Q1). Since the left hand side does not depend on \( D \) we conclude that \( v \in H^2(\omega) \). By exhausting \( \omega \) with \( D \ll \omega \) of regularity \( C^2 \) we have the claim.

3.5. Proof of Theorem 2.13.

Proof. The main point here is that we want to add the relaxation field \( \nabla h_n \vartheta_n \), that is given by the expression [74], to standard von Kármán type expansion. To make the procedure formally correct we would like to have that this relaxation field is bounded in \( L^\infty \). But we only can guarantee the equi-integrability of \( (\|\text{sym} \nabla h_n(k)\vartheta_k^2\|_{k \in \mathbb{N}}) \). We show, using the results from the Appendix, that we can replace these fields by bounded ones and then by diagonalization procedure approach asymptotic formula.

Without any loss of generality we can assume that \( \hat{R} = I \). First we assume that \( u \in C^1(\omega, \mathbb{R}^2), v \in C^2(\omega) \). The general claim will follow by density argument and by diagonalization, which is standard in \( \Gamma \)-convergence. Denote by \( M_1 = \text{sym} \nabla' u - \frac{1}{2} \vartheta' \nabla \otimes \nabla' v, M_2 = -\nabla^2 v \). By using Lemma 3.8 we find a subsequence, still denoted by \( (h_n)_{n \in \mathbb{N}} \) and \( (\varphi_n)_{n \in \mathbb{N}} \in H^2(\omega) \) and \( (\psi_n)_{n \in \mathbb{N}} \in H^1(\Omega, \mathbb{R}^3) \) such that such that \( \varphi_n \to 0 \) strongly in \( H^1 \), \( \psi_n \to 0 \) strongly in \( L^2 \). Moreover, the following holds
(a) \(|\nabla^2 \varphi_n|^2\)_{n \in \mathbb{N}} and \(|\nabla_h \psi_n|^2\)_{n \in \mathbb{N}} are equi-integrable,
(b) 
\[
\limsup_{n \to \infty} (\|\varphi_n\|_{H^2} + \|\nabla_h \psi_n\|_{L^2}) \leq C \left( \eta_2 \|M_1 + x_3 M_2\|^2_{L^2} + 1 \right).
\]
(c) For \((\vartheta_n)_{n \in \mathbb{N}} \subset H^1(\Omega; \mathbb{R}^3)\) defined by
\[
\vartheta_n = \psi_n + \begin{pmatrix} 0 \\ \phi_n \frac{x}{\varphi_n} \\ 3 \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n \\ \partial_2 \varphi_n \\ 0 \end{pmatrix},
\]
we have
\[
\int_\omega Q(x', M_1, M_2) \, dx' = \lim_{n \to \infty} \int_\Omega Q_{h_n}(x', \iota(M_1 + x_3 M_2) + \nabla_h \vartheta_n) \, dx.
\]
We know that \((\vartheta_n, 1, \vartheta_n, 2, h_n \vartheta_n, 3) \to 0\) strongly in \(L^2\) and that
\[
\text{sym} \nabla_h \vartheta_n = -x_3 \iota(\nabla^2 \varphi_n) + \text{sym} \nabla_h \psi_n,
\]
is \(L^2\) equi-integrable. In the same way as in the proof of Lemma 3.8 we can suppose that \(\varphi_n = \nabla \varphi_n = 0\) in a neighbourhood of \(\partial \omega\) and that \(\psi_n = 0\) in a neighbourhood of \(\partial \omega \times I\). We extend \(\varphi_n, \psi_n\) by zero on \(\tilde{\omega}\), where \(\tilde{\omega}\) has \(C^{1,1}\) boundary and \(\omega \subset \tilde{\omega}\). By using Corollary A.2 and Corollary A.4 we find for each \(\lambda > 0\) and \(n \in \mathbb{N}\), \(\varphi_n^\lambda \in H^2(\omega)\) and \(\psi_n^\lambda \in H^1(\Omega, \mathbb{R}^3)\) such that
\[
\sup_{n \in \mathbb{N}} \|\varphi_n^\lambda\|_{W^{2,\infty}} \leq C(\tilde{\omega}) \lambda,
\]
\[
\lim_{\lambda \to \infty} \sup_{n \in \mathbb{N}} \|\varphi_n^\lambda - \varphi_n\|_{H^2} = 0,
\]
\[
\sup_{\lambda > 0} \limsup_{n \to \infty} \|\varphi_n^\lambda\|_{H^2} \leq C(\tilde{\omega}) \left( \eta_2 \|M_1 + x_3 M_2\|^2_{L^2} + 1 \right),
\]
and
\[
\sup_{n \in \mathbb{N}} (\|\psi_n^\lambda\|_{L^\infty} + \|\nabla_h \psi_n^\lambda\|_{L^\infty}) \leq C(\tilde{\omega}) \lambda,
\]
\[
\lim_{\lambda \to \infty} \sup_{n \in \mathbb{N}} \left( \|\psi_n^\lambda - \psi_n\|_{L^2} + \|\nabla_h \psi_n^\lambda - \nabla_h \psi_n\|_{L^2} \right) = 0,
\]
\[
\sup_{\lambda > 0} \limsup_{n \to \infty} \left( \|\psi_n^\lambda\|_{L^2} + \|\nabla_h \psi_n^\lambda\|_{L^2} \right) \leq C(\tilde{\omega}) \left( \eta_2 \|M_1 + x_3 M_2\|^2_{L^2} + 1 \right).
\]
Notice that as the consequence of (77) and (80) we have
\[
\lim_{\lambda \to \infty} \limsup_{n \to \infty} \left( \|\varphi_n^\lambda\|_{H^1} + \|\psi_n^\lambda\|_{L^2} \right) = 0.
\]
Define
\[
\vartheta_n^\lambda = \psi_n^\lambda + \begin{pmatrix} 0 \\ \phi_n^\lambda \frac{x}{\varphi_n^\lambda} \\ 3 \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n^\lambda \\ \partial_2 \varphi_n^\lambda \\ 0 \end{pmatrix}.
\]
Again we have
\[
\text{sym} \nabla_h \vartheta_n^\lambda = -x_3 \iota(\nabla^2 \varphi_n^\lambda) + \text{sym} \nabla_h \psi_n^\lambda.
\]
Notice that due to (77), (80) we have
\[
\lim_{\lambda \to \infty} \sup_{n \in \mathbb{N}} \|\text{sym} \nabla_h \vartheta_n^\lambda - \text{sym} \nabla_h \vartheta_n\|_{L^2} = 0.
\]
Define also for every \( n, \lambda \) the function \( y_n^\lambda : \Omega \rightarrow \mathbb{R}^3 \) by

\[
y_n^\lambda(x', x_3) = \begin{pmatrix} x' \\ h_n x_3 \\ -\partial_1 (v + \varphi_n^\lambda)(x') - \partial_2 (v + \varphi_n^\lambda)(x') \end{pmatrix} - h_n x_3 \begin{pmatrix} \frac{1}{2} h_n^2 (v(x') + \varphi_n^\lambda(x')) \\ -\partial_1 (v + \varphi_n^\lambda)(x') \\ -\partial_2 (v + \varphi_n^\lambda)(x') \end{pmatrix} + h_n^2 \gamma_n^\lambda(x', x_3) + \begin{pmatrix} 0 \\ 0 \\ |\partial_1 (v + \varphi_n^\lambda)(x')| + |\partial_2 (v + \varphi_n^\lambda)(x')| \end{pmatrix}.
\]

From \( (82) \) we have

\[
\lim_{\lambda \to 0} \lim_{n \to \infty} \left( \left\| \frac{f y_n^\lambda - x'}{h_n^2} - u \right\|_{L^2} + \left\| \frac{f y_n^\lambda, 3}{h_n} - v \right\|_{L^2} \right) = 0,
\]

where \( y_n^\lambda = (y_{n,1}^\lambda, y_{n,2}^\lambda, y_{n,3}^\lambda) \). Also we easily conclude, by the Taylor expansion, that for every \( \lambda > 0 \)

\[
\lim_{n \to \infty} \left\| \frac{\sqrt{\nabla h_n \gamma_n^\lambda \nabla h_n \gamma_n^\lambda} - 1Id}{h_n^2} - E_n^\lambda \right\|_{L^2} = 0,
\]

where

\[
E_n^\lambda = \iota \left( \mathrm{sym} \nabla u - \frac{1}{2} \nabla (v + \varphi_n^\lambda) \otimes \nabla (v + \varphi_n^\lambda) - x_3 \nabla^2 v \right) + \mathrm{sym} \nabla h_n \theta_n.
\]

From property (W1), Lemma 3.11 and \( (5) \) we conclude that for every \( \lambda > 0 \) we have

\[
\lim_{n \to \infty} \left| \frac{1}{h_n^2} \int_{\Omega} W^{hn}(x, \nabla h_n y_n^\lambda) dx - \int_{\Omega} Q^{hn}(x, E_n^\lambda(x)) dx \right| = 0.
\]

Notice also that as a consequence of \( (77), (78), (82), (84) \) and the interpolation we have

\[
\lim_{\lambda \to 0} \sup_{n \to \infty} \left\| E_n^\lambda - E_n \right\|_{L^2} = 0,
\]

where

\[
E_n = \iota \left( \mathrm{sym} \nabla u - \frac{1}{2} \nabla v \otimes \nabla v - x_3 \nabla^2 v \right) + \mathrm{sym} \nabla h_n \theta_n.
\]

From \( (Q1), (5) \) and \( (74) \) we have

\[
\lim_{\lambda \to 0} \sup_{n \to \infty} \left| \int_{\Omega} Q^{hn}(x, E_n^\lambda(x)) dx - \int_{\omega} Q(x', M_1, M_2) dx' \right| = 0.
\]

By forming the function

\[
g(\lambda, n) = \left\| \frac{f y_n^\lambda - x'}{h_n^2} - u \right\|_{L^2(\omega)} + \left\| \frac{f y_n^\lambda, 3}{h_n} - v \right\|_{L^2(\omega)} + \left| \frac{1}{h_n^2} \int_{\Omega} W^{hn}(x, \nabla h_n y_n^\lambda) dx - \int_{\omega} Q(x', M_1, M_2) dx' \right|,
\]

we conclude from \( (85), (88) \) and \( (91) \) that

\[
\lim_{\lambda \to 0} \sup_{n \to \infty} g(\lambda, n) = 0.
\]

By performing diagonalizing argument we find monotone function \( \lambda(n) \), such that \( \lim_{n \to \infty} g(\lambda(n), n) = 0 \). This gives the desired sequence. To deal with \( u \in H^1(\omega, \mathbb{R}^3), v \in H^2(\omega) \) we need to do the further diagonalization. Namely, first we choose \( u_k \in C^1(\bar{\omega}, \mathbb{R}^3), v_k \in C^2(\bar{\omega}) \) such that

\[
\lim_{k \to \infty} \|u_k - u\|_{H^1} = 0, \quad \lim_{k \to \infty} \|v_k - v\|_{H^2} = 0.
\]

Denote by

\[
M_{1,k} = \mathrm{sym} \nabla' u_k - \frac{1}{2} \nabla' v_k \otimes \nabla' v_k, \quad M_{2,k} = -\nabla'^2 v_k.
\]
We have that $M_{1,k} \to M_1$, $M_{2,k} \to M_2$ strongly in $L^2$. Then for each $k \in \mathbb{N}$ we choose a sequence of functions $(y_{k,n})_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ such that
\[
\lim_{n \to \infty} \left( \left\| \frac{f_y y_{k,n} - x'}{h_n} - u_k \right\|_{L^2} + \left\| \frac{f_y y_{k,n}}{h_n} - v_k \right\|_{L^2} \right) = 0,
\]
and
\[
\lim_{n \to \infty} \left| \frac{1}{h_n^3} \int_{\Omega} W^{h_n}(x, \nabla y_{k,n}) \, dx - \int_{\omega} Q(x', M_{1,k}, M_{2,k}) \, dx' \right| = 0.
\]
From (Q’1) we see that
\[
\lim_{k \to \infty} \left| \int_{\omega} Q(x', M_{1,k}, M_{2,k}) \, dx' - \int_{\omega} Q(x', M_1, M_2) \, dx' \right| = 0.
\]
Thus, for the function formed by
\[
g(k, n) = \left( \left\| \frac{f_y y_{k,n} - x'}{h_n} - u \right\|_{L^2(\omega)} + \left\| \frac{f_y y_{k,n}}{h_n} - v \right\|_{L^2(\omega)} \right) + \left| \frac{1}{h_n^3} \int_{\Omega} W^{h_n}(x, \nabla y_{k,n}) \, dx - \int_{\omega} Q(x', M_1, M_2) \, dx' \right|
\]
we see that $\lim_{k \to \infty} \lim_{n \to \infty} g(k, n) = 0$. Then, by diagonalizing, we obtain the sequence $k(n)$ such that $\lim_{n \to \infty} g(k(n), n) = 0$. \qed

4. Periodic homogenization; the case of the oscillations in the direction of thickness

The previous result is given in terms of general non-periodic homogenization. In this section we will assume that we have periodic oscillations in the direction of thickness coupled together with the periodic oscillations in the in-plane directions. In [NV13] only periodic in-plane oscillations were considered.

We assume that the periodicity is given on two scales $\varepsilon_1(h)$ and $\varepsilon_2(h)$ where $\lim_{h \to 0} \varepsilon_1(h) = \lim_{h \to 0} \varepsilon_2(h) = 0$ and we denote $Y^d = [0, 1]^d$, where $d \in \{1, 2, 3\}$ for periodic cells. We denote by $\mathcal{Y}^d$ the sets $Y^d$ endowed with the torus topology. By $L^2(Y^d)$, $H^k(Y^d)$, $C(Y^d)$ we denote the subspaces of $L^2(Y^d)$, $H^k(Y^d)$, $C(Y^d)$ whose mid-value over $Y^d$ is zero. As usual, $C^k(Y^d)$ denotes the subspace of $C^k(Y^d)$ which consists of functions that are together with their $k$ derivatives $Y^d$-periodic, while $H^k(Y^d)$ denotes the closure of $C^k(Y^d)$ in $L^2$ norm. For $x \in \mathbb{R}^d$ by $(x)$ we denote $(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest element of $\mathbb{Z}^d$ less or equal to $x$.

We now give the assumptions on the energy densities with periodically oscillating material. We assume that a Borel measurable function $W : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty]$ satisfies
(a) $W(\cdot, F)$ is $Y^3$-periodic for all $F \in \mathbb{R}^{3 \times 3}$;
(b) there exists $\eta_1, \eta_2, \rho > 0$ such that for a.e. $y \in Y^3$, we have that $W(y, \cdot) \in W(\eta_1, \eta_2, \rho)$;
(c) there exists $\tilde{Q} : Y^3 \times \mathbb{R}^{3 \times 3} \to [0, \infty)$ such that, for a.e. $y \in Y^3$, $\tilde{Q}(y, \cdot)$ is a quadratic form and the following holds
\[(92) \quad \forall G \in \mathbb{R}^{3 \times 3}, \quad \text{ess sup}_{y \in Y^3} |W(y, Id + G) - \tilde{Q}(y, G)| \leq r(|G|)|G|^2,
\]
where $r : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is a monotone function that satisfies $\lim_{\delta \to 0} r(\delta) = 0$. 

It is easy to see that for a.e. $y \in \mathbb{Y}^3$, $\bar{Q}(y, \cdot)$ is a quadratic form given in the Definition 2.1 and that for a.e. $y \in [0, 1)^3$ we have

$$\eta_1|\text{sym } A|^2 \leq \bar{Q}(y, A) = \bar{Q}(y, \text{sym } A) \leq \eta_2|\text{sym } A|^2, \forall A \in \mathbb{R}^{3 \times 3}.$$  

**Definition 4.1** (Two-scale convergence with respect to two scales $\varepsilon_1$ and $\varepsilon_2$). We say that a bounded sequence $(u^h)_{h>0}$ in $L^2(\Omega)$ two-scale converges to $u \in L^2(\Omega \times \mathbb{Y}^3)$ and we write $u^h(x) \overset{2}{\rightharpoonup} u(x, y)$ if

$$\lim_{h \to 0} \int_{\Omega} u^h(x) \psi \left( x, \frac{x'}{\varepsilon_1(h)}, \frac{x_3}{\varepsilon_2(h)} \right) dx = \int_{\Omega \times \mathbb{Y}^3} u(x, y) \psi(x, y) dy dx,$$

for all $\psi \in C_0^\infty(\Omega, C(\mathbb{Y}^3))$. If, in addition, $\|u^h\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega \times \mathbb{Y}^3)}$ we say that $u^h$ strongly two-scale converges to $u$ and write $u^h \rightharpoonup^s u$. For vector-valued functions, two-scale convergence is defined componentwise.

It can be proved that all standard claims for two-scale convergence (such as compactness, lower semicontinuity of convex integrands) also holds in this case (for two-scale convergence see [All92, Vis06, Vis07]). Here we study two regimes in the periodic context, which we find the most interesting.

1. $\varepsilon_1(h) = h^{\alpha+1}$ and $\varepsilon_2(h) = h^{\alpha}$ for some $\alpha > 0$.
2. $\lim_{h \to 0} \frac{h}{\varepsilon_1(h)} = 1$ and $\lim_{h \to 0} \varepsilon_2(h) = 0$.

We define the energy functionals $I^h(y^h) : W^{1,2}(\Omega; \mathbb{R}^3) \to \mathbb{R}$ by

$$I^h(y^h) = \int_{\Omega} W \left( \frac{x'}{\varepsilon_1(h)}, \frac{x_3}{\varepsilon_2(h)}, \nabla_h y^h \right) dx.$$

To prove the cell formula we will need the following proposition which is given in [MVa].

**Proposition 4.2.** Let $(u^h)_{h>0}$ be a weakly convergent sequence in $H^1(\Omega, \mathbb{R}^3)$ with limit $u^0$ and assume that $(\|\nabla_h u^h\|_{L^2(\Omega)})_{h>0}$ is bounded.

1. Suppose that $\varepsilon_1(h) = h^{\alpha+1}$ and $\varepsilon_2(h) = h^{\alpha}$ for some $\alpha > 0$. Then $u^0$ is independent of $x_3$ and there are functions $u^1 \in L^2(\omega; H^1(I; \mathbb{R}^3))$ and $u^2 \in L^2(\Omega; H^1(\mathbb{Y}^3; \mathbb{R}^3))$ such that

$$\nabla_h u^h \overset{2}{\rightharpoonup} (\nabla' u^0 | \partial_{x^3} u^1) + \nabla_y u^2,$$

weakly two-scale in $L^2(\Omega \times \mathbb{Y}^3; \mathbb{R}^3)$ on a subsequence.

2. Suppose that $\lim_{h \to 0} \frac{h}{\varepsilon_1(h)} = 1$ and $\lim_{h \to 0} \varepsilon_2(h) = 0$. Then $u^0$ is independent of $x_3$ and there are functions $u^1 \in L^2(\Omega; H^1(\mathbb{Y}^2; \mathbb{R}^3))$ and $d \in L^2(\Omega \times \mathbb{Y}^3; \mathbb{R}^3)$ such that $\int_Y d(x, y', y_3) dy_3 = 0$ for a.e. $(x, y') \in \Omega \times \mathbb{Y}^2$ and

$$\nabla_h u^h \overset{2}{\rightharpoonup} (\nabla' u^0 | 0) + (\nabla_y u^1 | \partial_{x^3} u^1) + (0 | 0 | d),$$

weakly two-scale in $L^2(\Omega \times \mathbb{Y}^3; \mathbb{R}^3)$ on a subsequence.

The following proposition gives us the cell formula.

**Proposition 4.3.** Let $W : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to [0, \infty]$ be a Borel measurable function that satisfies (a)-(c) and let $W^h(x, \cdot) = W \left( \frac{x'}{\varepsilon_1(h)}, \frac{x_3}{\varepsilon_2(h)} \right)$. Then for every subsequence $(h_n)_{n \in \mathbb{N}}$ monotonically decreasing to zero we have that (i), (ii), (iii) of Theorem 2.6 is satisfied. Moreover the limit energy density is a homogeneous quadratic functional $Q : \mathbb{R}^{3 \times 2} \times \mathbb{R}^{2 \times 2} \to \mathbb{R}$ given by
1. \( (\varepsilon_1(h) = h^{\alpha+1}, \varepsilon_2(h) = h^{\alpha} ) \)

\[
Q(M_1, M_2) := \inf_{d} \int_{I} \left( \inf_{\phi} \int_{Y_3} Q(y, \iota(M_1 + x_3 M_2) + (0|0|d) + \nabla_y \phi) \, dy \right) \, dx_3,
\]

where the infimum is taken over \( d \in L^2(I; \mathbb{R}^3) \) and \( \phi \in H^1(Y^3; \mathbb{R}^3) \).

2. \( (\lim_{h \to 0} \frac{\varepsilon_1(h)}{h} = 1, \lim_{h \to 0} \varepsilon_2(h) = 0) \)

\[
Q(M_1, M_2) := \inf_{\phi} \int_{I \times Y^2} \left( \int_{d} \int_{Y^3} \tilde{Q}(y, \iota(M_1 + x_3 M_2) + (\nabla_y \phi, \partial_3 \phi) + (0|0|d)) \, dy_3 \right) \, dy \, dx_3,
\]

where the infimum is taken over \( \phi \in H^1(I \times Y^2; \mathbb{R}^3) \), \( d \in L^2(Y^1; \mathbb{R}^3) \).

**Proof.** Take a subsequence \( (h_n)_{n \in \mathbb{N}} \) monotonically decreasing to zero such that the Assumption \( 2.8 \) is satisfied. We will only prove the case when \( \varepsilon_1(h) = h^{\alpha+1}, \varepsilon_2(h) = h^{\alpha} \). Notice that \( Q \) can be written in the following way

\[
Q(M_1, M_2) := \inf_{\phi,d} \int_{Y^3 \times I} Q(y, \iota(M_1 + x_3 M_2) + (0|0|d) + \nabla_y \phi) \, dy \, dx_3,
\]

where the infimum is taken over \( d \in L^2(I; \mathbb{R}^3) \), \( \phi \in L^2(I; \hat{H}^1(Y^3; \mathbb{R}^3)) \). Moreover, by the direct methods of the calculus of variation, the infimum is attained for some \( d_M \in L^2(I; \mathbb{R}^3) \), \( \phi_M \in L^2(I; \hat{H}^1(Y^3; \mathbb{R}^3)) \) and the following inequality holds

\[
|d_M|^2_{L^2(I; \mathbb{R}^3)} + ||\phi_M||^2_{L^2(I; \hat{H}^1(Y^3; \mathbb{R}^3))} \leq C(\eta_1, \eta_2)(|M_1|^2 + |M_2|^2),
\]

for some \( C(\eta_1, \eta_2) > 0 \) dependent only on \( \eta_1, \eta_2 \). We know from Proposition \( 2.10 \) that for \( x_0' \in \omega \) we have

\[
Q(x_0', M_1, M_2) = \lim_{r \to 0} \frac{1}{|B(x_0', r)|} K(M_1 + x_3 M_2, B(x_0', r)).
\]

By using the definition of the functional \( K(\cdot, \cdot) \), as well as Proposition \( 3.3 \), we conclude in the same way as in the proof of Lemma \( 3.5 \) that there exist \( (\varphi^{h_n})_{n \in \mathbb{N}} \subset H^2(B(x', r)) \), bounded in \( H^2, \varphi^{h_n} \to 0 \) strongly in \( H^1 \), and \( (\psi^{h_n})_{n \in \mathbb{N}} \subset H^1(B(x', r) \times I; \mathbb{R}^3) \), \( \psi^{h_n} \to 0 \) strongly in \( L^2 \), such that \( (\nabla h_n \psi^{h_n})_{n \in \mathbb{N}} \) is bounded in \( L^2 \) and we have

\[
K(M_1 + x_3 M_2, B(x_0', r)) = \lim_{n \to \infty} \int_{B(x_0', r) \times I} \tilde{Q} \left( \left( \frac{x_0'}{h_n}, \frac{x_0'}{h_n} \right), \iota(M_1 + x_3 M_2) + x_3 \iota(\nabla^2 \varphi^{h_n}) + \nabla h_n \psi^{h_n} \right) dx' \, dx_3.
\]

Using Proposition \( 4.2 \) we conclude that there exist \( d \in L^2(I; \mathbb{R}^3) \) and \( \phi \in L^2(\Omega; \hat{H}^2(Y^2)) \) such that

\[
\nabla h_n \psi^{h_n} \rightarrow^* (0|0|d) + \nabla_y \tilde{\phi},
\]

weakly two-scale in \( L^2(\Omega \times Y^3; \mathbb{R}^3) \) on a subsequence. By using [Vel13 Lemma 3.8] we conclude that there exists \( \xi \in L^2(\omega; \hat{H}^2(Y^2)) \) such that

\[
\nabla^2 \psi^{h_n} \rightarrow^* \nabla^2 \xi.
\]

After putting \( \phi = \tilde{\phi} + x_3(\partial_y \xi, \partial_y \xi, 0) \) we obtain that

\[
x_3 \iota(\nabla^2 \psi^{h_n}) + \nabla h_n \psi^{h_n} \rightarrow^* (0|0|d) + \nabla_y \tilde{\phi}.
\]

Applying lower semicontinuity of the integral functionals with periodic, convex integrands (see [Vis06, Vis07]) with respect to two scale convergence we conclude

\[
K(M_1 + x_3 M_2, B(x_0', r)) \geq |B(x_0', r)| Q(M_1, M_2).
\]
To obtain the opposite inequality we construct the relaxation sequence of the type
\[ \psi_{h_n}(x) = h_n \int_0^{x_3} \hat{d}_M(t) dt + h_n^{\alpha+1} \hat{\phi}_M \left( x_3, \left\langle \frac{x'}{h_n} \right\rangle, \left\langle \frac{x_3}{h_n} \right\rangle \right), \]
for some \( \hat{d}_M \in C(I; \mathbb{R}^3), \hat{\phi}_M \in C^1(I; C^1(Y^3; \mathbb{R}^3)) \). From the definition of \( K(\cdot, \cdot) \), using the inequality (94), smooth approximation and the continuity of integral functionals with periodic integrands with respect to strong two-scale convergence we obtain the opposite inequality
\[ K(M_1 + x_3 M_2, B(x'_0, r)) \leq |B(x'_0, r)| Q(M_1, M_2). \]
From this we have the claim. The case 2 can be treated in the similar way. The important fact is to notice that for \( \xi \in H^2(Y^2) \) we have
\[ x_3 \iota(\nabla^2 y \xi) = \text{sym} \left( \nabla^\prime \tilde{\phi} |_{\partial x_3} \tilde{\phi} \right), \]
where \( \tilde{\phi} = (x_3 \partial_1 \xi, x_3 \partial_2 \xi, -\xi)' \), and, thus, again this member can be absorbed as in the case 1. \( \square \)

**Remark 9.** It was shown in [NV13] that, in the case of periodic in-plane oscillations in the regime when the thickness is significantly less than the period of the oscillations, we can not neglect the contribution of the sequence \( \nabla^2 \varphi_{h_n} \).

**Appendix A. Auxiliary results**

**Proposition A.1.** Let \( 1 \leq p \leq \infty, \lambda > 0 \). Let \( A \) be a bounded open set in \( \mathbb{R}^n \) with Lipschitz boundary

(a) Suppose \( u \in W^{1,p}(A) \) then there exists \( u^\lambda \in W^{1,\infty}(A) \) such that
\[
\|u^\lambda\|_{W^{1,\infty}} \leq C(n, p, A) \lambda \left\{ \int_{\{ |u| + |\nabla u| \geq \lambda/C(n, p, A) \}} (|u| + |\nabla u|)^p dx \right\}. \]
In particular,
\[
\lim_{\lambda \to \infty} \lambda^p \left\{ \{ x \in A : u^\lambda(x) \neq u(x) \} \right\} = 0. \]
If we define Hardy Littlewood maximal function
\[ M_a(x) = \sup_{r>0} \int_{B(x, r)} a(y) dy, \]
where \( a = |\tilde{u}| + |\nabla \tilde{u}| \) (\( \tilde{u} \) is the extension of \( u \) to \( W^{2,2}(\mathbb{R}^n) \) which has the compact support) and
\[ A^\lambda = \{ x \in \mathbb{R}^n : M_a(x) < \lambda \text{ and } x \text{ is a Lebesgue point of } u, \nabla u \text{ and } \nabla^2 u \}, \]
then we can construct \( u^\lambda \) such that
\[ \{ u^\lambda \neq u \} = \tilde{A}^\lambda, \]
where \( \tilde{A}^\lambda \) is a closed subset of \( A^\lambda \cap A \) which satisfies \( |A \setminus \tilde{A}^\lambda| \leq C |A \setminus A^\lambda| \), for some \( C > 1 \).
(b) Assume additionally that $A$ is has the boundary of class $C^{1,1}$ and $u \in W^{2,p}(A)$.
Then there exists $u^\lambda \in W^{2,\infty}(A)$ such that
\[
|u^\lambda|_{W^{2,\infty}} \leq C(n,p,A)\lambda,
\]
\[
|\{ x \in A : u^\lambda(x) \neq u(x) \}| \leq C(n,p,A) \int \frac{1}{\lambda^p} \int_{\{(|u|+|\nabla u|+|\nabla^2 u|) \geq \lambda/C(n,p,A) \}} (|u| + |\nabla u| + |\nabla^2 u|)^p \, dx,
\]
where $a = |u| + |\nabla u| + |\nabla^2 u|$. If we define
\[
Ma(x) = \sup_{r > 0} \int_{B(x,r)} a(y) \, dy,
\]
and
\[
A^\lambda = \{ x \in A : Ma(x) < \lambda \text{ and } x \text{ is a Lebesgue point of } u, \nabla u \text{ and } \nabla^2 u \},
\]
then we can construct $u^\lambda$ such that
\[
\{ u^\lambda \neq u \} = \bar{A}^\lambda,
\]
where $\bar{A}^\lambda$ is a closed subset of $A^\lambda$ which satisfies $|A \setminus \bar{A}^\lambda| \leq C|A \setminus A^\lambda|$, for some $C > 1$.

Proof. See the proof of Proposition A2 in [FJM02]. The condition in (b) that the domain is of class $C^{1,1}$ is not demanded there. The argument is that one can extend $W^{2,p}(A)$ to $W^{2,p}(\mathbb{R}^n)$ when $A$ is only Lipschitz (see e.g. [Ste70]). However, if for $u \in W^{2,p}(A)$ we denote this extension by $Eu$ then it is not clear to the author weather the term
\[
\int_{\{(|Eu|+|\nabla Eu|+|\nabla^2 Eu|) \geq \lambda/C_1(n,p,A) \}} (|Eu| + |\nabla Eu| + |\nabla^2 Eu|)^p \, dx,
\]
can be controlled with the term
\[
\int_{\{(|u|+|\nabla u|+|\nabla^2 u|) \geq \lambda/C_2(n,p,A) \}} (|u| + |\nabla u| + |\nabla^2 u|)^p \, dx.
\]
For the standard extension operator, constructed using the reflexion, this can be easily proved to be valid. □

Remark 10. Notice that due to [EG92, Theorem 3, Section 6] for $u, v \in W^{1,p}(A)$ we have that
\[
\{ u = v \} = \{ u = v, \nabla u = \nabla v \} \cup N,
\]
where $N$ is the set of measure zero. From this it follows that for $u, v \in W^{2,p}(A)$ we have that
\[
\{ u = v \} = \{ u = v, \nabla u = \nabla v, \nabla^2 u = \nabla^2 v \} \cup N,
\]
where $N$ is the set of measure zero.

Corollary A.2. Let $1 \leq p \leq \infty$, $\lambda > 0$ and let $A$ be a open bounded open set in $\mathbb{R}^n$ with Lipschitz boundary.

(a) Suppose that $(u^h)_{h>0} \subset W^{1,p}(A)$ is a sequence such that $u^h \rightharpoonup u$ weakly in $W^{1,p}$ and $|\nabla u^h|^p$ is equi-integrable. Then there exists $(u^{\lambda,h})_{\lambda,h>0}$ such that
\[
\|u^{\lambda,h}\|_{W^{1,\infty}} \leq C(n,p,A)\lambda,
\]
\[
\limsup_{\lambda \to \infty} \|u^{\lambda,h} - u^h\|_{W^{1,p}} = 0,
\]
\[
\|u^{\lambda,h}\|_{W^{1,p}} \leq C(n,p,A)\|u^h\|_{W^{1,p}}.
\]
For each \( u \)

**Proof.** The proof is the direct consequence of Proposition \[A.1\]. We will prove only (a).

For each \( u \) and \( \lambda > 0 \) we choose \( u^{\lambda,h} \) such that

\[
\|u^{\lambda,h}\|_{W^{1,\infty}} \leq C(n,p,A)\lambda, \\
\lim_{\lambda \to \infty} \sup_{h>0} \|u^{\lambda,h} - u^h\|_{W^{2,p}} = 0, \\
\|u^{\lambda,h}\|_{W^{2,p}} \leq C(n,p,A)\|u^h\|_{W^{2,p}}.
\]

(b) Assume additionally that \( A \) has the boundary of class \( C^{1,1} \) and that \( (u^h)_{h>0} \subset W^{2,p}(A) \) is a sequence such that \( u^h \rightharpoonup u \) weakly in \( W^{2,p} \) and \( (\|\nabla^2 u^h\|)_{h>0} \) is equi-integrable. Then there exists \( (u^{\lambda,h})_{\lambda,h>0} \) such that

\[
\|u^{\lambda,h}\|_{W^{1,\infty}} \leq C(n,p,A)\lambda, \\
\lim_{\lambda \to \infty} \sup_{h>0} \|u^{\lambda,h} - u^h\|_{W^{2,p}} = 0, \\
\|u^{\lambda,h}\|_{W^{2,p}} \leq C(n,p,A)\|u^h\|_{W^{2,p}}.
\]

**Proof.** The proof is the direct consequence of Proposition \[A.1\]. We will prove only (a).

For each \( u \) and \( \lambda > 0 \) we choose \( u^{\lambda,h} \) such that

\[
\|u^{\lambda,h}\|_{W^{1,\infty}} \leq C(n,p,A)\lambda, \\
|A^{\lambda,h}| \leq \frac{C(n,p,A)}{\lambda^p} \int_{\{u^h + |\nabla u^h| \geq \lambda/C(n,p,A)\}} (|u^h| + |\nabla u^h|)^p \, dx,
\]

where \( A^{\lambda,h} = \{ x \in A : u^h(x) \neq u(x) \} \). Notice that since \( u^h \rightharpoonup u \) strongly in \( L^p \) and \( (\|\nabla^2 u^h\|)_{h>0} \) is equi-integrable we have that

\[
\lim_{\lambda \to \infty} \sup_{h>0} \int_{\{u^h + |\nabla u^h| \geq \lambda/C(n,p,A)\}} (|u^h| + |\nabla u^h|)^p \, dx = 0.
\]

From this we easily see that \( \lim_{\lambda \to \infty} \sup_{h>0} \lambda^p |A^{\lambda,h}| = 0 \). Using \[95\] we conclude that

\[
\lim_{\lambda \to \infty} \sup_{h>0} \left( \|u^h\|_{L^p(A^{\lambda,h})} + \|\nabla u^h\|_{L^p(A^{\lambda,h})} \right) \to 0, \\
\lim_{\lambda \to \infty} \sup_{h>0} \left( \|u^{\lambda,h}\|_{L^p(A^{\lambda,h})} + \|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})} \right) \to 0.
\]

Notice also simple estimate

\[
\|u^{\lambda,h}\|_{L^p(A^{\lambda,h})} + \|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})} \leq 2C(n,p,A)^2\|u^h\|_{W^{1,p}},
\]

From this we have the claim since

\[
\|u^{\lambda,h} - u^h\|_{W^{1,p}} = \|u^{\lambda,h} - u^h\|_{L^p(A^{\lambda,h})} + \|\nabla u^{\lambda,h} - \nabla u^h\|_{L^p(A^{\lambda,h})}, \\
\|u^{\lambda,h}\|_{W^{1,p}} = \|u^h\|_{L^p(A^{\lambda,h}^c)} + \|\nabla u^h\|_{L^p(A^{\lambda,h}^c)} + \|u^{\lambda,h}\|_{L^p(A^{\lambda,h})} + \|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})},
\]

The following proposition we prove by combining the ideas of extension given in \[BF02\] and \[BZ07\] with Proposition \[A.1\].

**Proposition A.3.** Let \( 1 \leq p \leq \infty \) and \( A \) be a bounded open set in \( \mathbb{R}^2 \) with Lipschitz boundary. Suppose that \( u \in W^{1,p}(A \times I, \mathbb{R}^3) \). Then for every \( 0 < h < 1 \) there exists \( u^{\lambda,h} \in W^{1,\infty}(A \times I, \mathbb{R}^3) \) such that

\[
\|u^{\lambda,h}\|_{L^\infty} + \|\nabla u^{\lambda,h}\|_{L^\infty} \leq C(n,p,A)\lambda, \\
|\{ x \in A : u^{\lambda,h}(x) \neq u(x) \}| \leq \frac{C(n,p,A)}{\lambda^p} \int_{\{u + |\nabla u| \geq \lambda/C(n,p,A)\}} (|u| + |\nabla u|)^p \, dx.
\]
Proof. The idea is to look the problem on the physical domain $A \times hI$, extend it by reflection and translation to the domain $A \times I$ and then apply Proposition A.1 and choose the good strip. Define $\tilde{u} : A \times I \to \mathbb{R}^3$ as $2h$ periodic function in the variable $x_3$ in the following way

$$\tilde{u}^h(x', x_3) = \begin{cases} u(x', \frac{x_3}{h}), & \text{if } x_3 \in hI, \\ u(x', 1 - \frac{x_3}{h}), & \text{if } x_3 \in [h/2, 3h/2], \end{cases}$$

e and extend it by periodicity on $A \times I$. This implies that we have $2l + 1 = 2[\frac{1}{2h} - \frac{1}{2}] + 1$ whole strips and at most 2 strips with the boundary $x_3 = 1/2$ i.e. $x_3 = -1/2$ where the function $\tilde{u}^h$ does not exhaust the full period $2h$. Denote for $i \in \{-l, \ldots, l\}$ the sets $K_i = [(2i - 1)h/2, (2i + 1)h/2]$ and $L_1 = [(2l + 1)h/2, 1/2]$, $L_2 = [-1/2, -(2l + 1)h/2]$. Notice that $I = \bigcup_{i=0}^{l} K_i \cup L_1 \cup L_2$ and

$$\nabla \tilde{u}^h = \begin{cases} \nabla_h u(x', \frac{x_3}{h}), & \text{if } x_3 \in hI, \\ (\partial_1 u(x', 1 - \frac{x_3}{h}) , \partial_2 u(x', 1 - \frac{x_3}{h}), -\frac{1}{h} \partial_3 u(x', 1 - \frac{x_3}{h})) , & \text{if } x_3 \in [h/2, 3h/2]. \end{cases}
$$

Notice that $\tilde{u}^h \in W^{1,p}(A \times I, \mathbb{R}^3)$. We apply Proposition A.1 on the function $\tilde{u}^h$ to obtain the function $\tilde{u}^\lambda$ such that

$$\|\tilde{u}^\lambda\|_{W^{1,\infty}} \leq C(n, p, A) \lambda,$$

$$|\{x \in A : \tilde{u}^\lambda(x) \neq \tilde{u}(x)\}| \leq \frac{C(n, p, A)}{\lambda^p} \int_{\|\tilde{u}^h\| + \|\nabla \tilde{u}^h\| \geq \lambda / C(n, p, A)} (|\tilde{u}^h| + |\nabla \tilde{u}^h|)^p\, dx.$$

We want to show that there exists strip which satisfies the appropriate estimate. Notice that, due to our construction we have for every $i \in \{-l, \ldots, l\}$ and $j \in \{1, 2\}$

$$\int_{\|\tilde{u}^h\| + \|\nabla \tilde{u}^h\| \geq \lambda / C(n, p, A)} (|\tilde{u}^h| + |\nabla \tilde{u}^h|)^p\, dx =$$

$$h \int_{\|u\| + \|\nabla u\| \geq \lambda / C(n, p, A)} (|u| + |\nabla u|)^p\, dx,$$

$$\int_{\|\tilde{u}^h\| + \|\nabla \tilde{u}^h\| \geq \lambda / C(n, p, A)} (|\tilde{u}^h| + |\nabla \tilde{u}^h|)^p\, dx$$

$$\leq h \int_{\|u\| + \|\nabla u\| \geq \lambda / C(n, p, A)} (|u| + |\nabla u|)^p\, dx.$$

From this we conclude that there exists strip i.e. $i \in \{-l, \ldots, l\}$ and the set $A \times K_i$ such that

$$\frac{1}{h}|\{x \in A \times K_i : \tilde{u}^\lambda(x) \neq \tilde{u}(x)\}| \leq \frac{3C(n, p, A)}{\lambda^p} \int_{\|u\| + \|\nabla u\| \geq \lambda / C(n, p, A)} (|u| + |\nabla u|)^p\, dx.$$

To obtain $u^\lambda$ we take $\tilde{u}^\lambda|_{A \times K_i}$, translate it to the strip $A \times [-h/2, h/2]$, if necessary reflect it, and then stretch it to the domain $A \times I$. \hfill \square

The proof of the following corollary goes in the same way as the proof of Corollary A.2 using Proposition A.3 instead of Proposition A.1. We will just state the result.

**Corollary A.4.** Let $1 \leq p \leq \infty$ and $A$ be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. Suppose that $(u^h)_{h>0} \subset W^{1,p}(A)$ is a sequence such that $u^h \rightharpoonup u$ weakly in
\[ W^{1,p} \text{ and } (|\nabla_h u^h|^p)_{h>0} \text{ is equi-integrable. Then there exists } (u^{\lambda,h})_{\lambda,h>0} \text{ such that} \]
\[ \|u^{\lambda,h}\|_{L^\infty} + \|\nabla_h u^{\lambda,h}\|_{L^\infty} \leq C(n,p,A)\lambda, \]
\[ \lim_{\lambda \to \infty} \sup_{h>0} \left( \|u^{\lambda,h} - u^h\|_{L^p} + \|\nabla_h u^{\lambda,h} - \nabla_h u^h\|_{L^p} \right) = 0, \]
\[ \|u^{\lambda,h}\|_{L^p} + \|\nabla_h u^{\lambda,h}\|_{L^p} \leq C(n,p,A) \left( \|u^h\|_{L^p} + \|\nabla_h u^h\|_{L^p} \right). \]

The following proposition is just simple adaption of [FMP98, Lemma 1.2].

**Proposition A.5.** Let \( p > 1 \). Let \( A \subset \mathbb{R}^n \) be a open bounded set.

(a) Let \((w_n)_{n \in \mathbb{N}}\) be a bounded sequence in \( W^{1,p}(A) \). Then there exist a subsequence \((w_{n(k)})_{k \in \mathbb{N}}\) and a sequence \((z_k)_{k \in \mathbb{N}} \subset W^{1,p}(A)\) such that
\[ \{z_k \neq w_{n(k)}\} \to 0, \]
as \( k \to \infty \) and \( (|\nabla z_k|^p)_{k \in \mathbb{N}} \) is equi-integrable. Each \( z_k \) may be chosen to be Lipschitz function. If \( w_n \rightharpoonup w \) weakly in \( W^{1,p} \) then \( z_k \rightharpoonup w \) weakly in \( W^{1,p} \).

(b) Let \((w_n)_{n \in \mathbb{N}}\) be a bounded sequence in \( W^{2,p}(A) \). Then there exist a subsequence \((w_{n(k)})_{k \in \mathbb{N}}\) and a sequence \((z_k)_{k \in \mathbb{N}} \subset W^{2,p}(A)\) such that
\[ \{z_k \neq w_{n(k)}\} \to 0, \]
as \( k \to \infty \) and \( (|\nabla^2 z_k|^p)_{k \in \mathbb{N}} \) is equi-integrable. Each \( z_k \) may be chosen such that \( z_k \in W^{2,\infty}(S) \). If \( w_n \rightharpoonup w \) weakly in \( W^{2,p} \) then \( z_k \rightharpoonup w \) weakly in \( W^{2,p} \).

**Proof.** Proof of (i) is given in [FMP98, Lemma 1.2]. The proof of (ii) goes in the same way. We can assume that the boundary of \( A \) is of class \( C^{1,1} \) (to deal with general open bounded set see the proof of Step 2 in [FMP98, Lemma 1.2]. Namely, we extend each \( w_n \) on \( \mathbb{R}^n \) such that the support of each \( w_n \) lies in a fixed compact subset \( K \subset \mathbb{R}^n \). We denote this extension also by \( w_n \). Denote by \( a_n = |w_n| + |\nabla w_n| + |\nabla^2 w_n| \) and by
\[ M_\alpha(x) = \sup_{r>0} \int_{B(x,r)} a_n(y)dy, \]
the Hardy Littlewood maximal function. It is well known that
\[ \|M(a_n(x))\|_{L^p(\mathbb{R}^n)} \leq C(n,p\|w_n(\cdot)\|_{W^{1,p}(\mathbb{R}^n)} \leq C(n,p,A)\|w_n(\cdot)\|_{W^{2,p}(A)} \]
We denote by \( \mu = \{\mu_x\}_{x \in \Omega} \) the Young measures associated with the converging subsequence of \( (M(\nabla a_n))_{n \in \mathbb{N}} \). We have the following properties
\[(a) \int_\Omega \int_{\mathbb{R}} |s|^p d\mu_x < +\infty. \]
\[(b) \text{ whenever } (f(M(a_n)))_{n \in \mathbb{N}} \text{ converges weakly in } L^1(\Omega), \text{ its weak limit is given by} \]
\[ \bar{f}(x) := \langle \mu_x, f \rangle, \text{ a.e. } x \in \Omega. \]

For \( k \in \mathbb{N} \) we consider the truncation map \( T_k : \mathbb{R} \to \mathbb{R} \) given by
\[ T_k := \left\{ x, \begin{array}{ll} x, & |x| \leq k, \\ \frac{x}{|x|}, & |x| > k \end{array} \right. . \]

In the same way as in proof of Lemma [FMP98, Lemma 1.2] we obtain a subsequence \((w_{n(k)})_{k \in \mathbb{N}}\) such that
\[ |T_k(M(a_{n(k)}))|^p \rightharpoonup f \text{ weakly in } L^1(A), \]
where
\[ f(x) = \int_{\mathbb{R}} |s|^p d\mu_x(s). \]
Set
\[ \tilde{R}_k := \{ x \in \mathbb{R}^n : M(a_{n(k)}) < k \}. \]
Notice that for \( k \) large enough, since the support of \( w_{n(k)} \) lies in \( K \), we have that \( \tilde{R}_k \subset K_1 \) where \( K_1 \) is a compact subset of \( \mathbb{R}^n \), \( K_1 \supset K \). So without the loss of generality we can assume that for each \( k \) we have \( \tilde{R}_k \subset K_1 \). Denote by \( R_k \) the closed subset of \( \tilde{R}_k \cap A \) such that
\[ |A \setminus \tilde{R}_k| \leq 2|A \setminus R_k|, \]
and
\[ |\tilde{R}_k \setminus R_k| \leq \frac{1}{kp+1}. \]
By Proposition A.1(ii) there exists \( z_k \in W^{2,\infty}(A) \) such that
\[ z_k = w_{n(k)} \text{ a.e. on } R_k, \quad \|z_k\|_{W^{2,\infty}} \leq C(n, p, A)k. \]
We have
\[ |\{ x \in \Omega : z_k \neq w_{n(k)} \}| \leq |\tilde{R}_k| + \frac{1}{kp+1} \leq \frac{1}{kp} \|Ma_{n(k)}\|_{L^p} + \frac{1}{kp+1}, \]
and this term tends to zero as \( k \to \infty \). For a.e. \( x \in R_k \) we have
\[ |\nabla^2 z_k| = |\nabla^2 w_{n(k)}| \leq |M(a_{n(k)})| = |T_k(M(a_{n(k)}))|, \]
while if \( x \in A \cap \tilde{R}_k \) we have
\[ |\nabla^2 z_k(x)| \leq C(n, p, A)k \leq C(n, p, A) |T_k(M(a_{n(k)}))(x)|. \]
For \( x \in (A \cap \tilde{R}_k) \setminus R_k \) we can only conclude
\[ |\nabla^2 z_k(x)| \leq C(n, p, A)k. \]
Since we have
\[ \int_A |\nabla^2 z_k| dx = \int_{R_k} |\nabla^2 z_k| dx + \int_{A \setminus R_k} |\nabla^2 z_k| dx + \int_{(A \cap \tilde{R}_k) \setminus R_k} |\nabla^2 z_k| dx. \]
and
\[ \int_{R_k} |\nabla^2 z_k|^p dx \leq \int_{R_k} |T_k(M(a_{n(k)}))^p|, \]
\[ \int_{A \setminus R_k} |\nabla^2 z_k|^p \leq \int_{A \setminus R_k} |T_k(M(a_{n(k)}))^p|, \]
\[ \int_{(A \cap \tilde{R}_k) \setminus R_k} |\nabla^2 z_k| dx \leq \frac{C(n, p, A)}{k}, \]
taking into account that \( T_k(M(a_{n(k)})) \) is equi-integrable, we have the claim. It is easy to see that from the property \([100]\) it follows that \((z_k)_{k \in \mathbb{N}}\) has the same weak limit as \((w_k)_{k \in \mathbb{N}}\). \( \square \)

The following proposition can be found in [BF02] (see also [BZ07]).

**Proposition A.6.** Let \( 1 < p < +\infty \) and \( A \subset \mathbb{R}^2 \) be a open bounded set with Lipschitz boundary. Let \((h_n)_{n \in \mathbb{N}} \) be a sequence of positive numbers converging to zero and let \((w_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( W^{1,p}(A \times I, \mathbb{R}^3) \) satisfying:
\[ \limsup_{n \in \mathbb{N}} \int_{A \times I} \left( \partial_1 w_n \partial_2 w_n \frac{1}{h_n} \partial_3 w_n \right)^p dx < +\infty. \]
Suppose further that \( w_n \rightharpoonup \psi \) weakly in \( W^{1,p}(A \times I, \mathbb{R}^3) \). Then there exists a subsequence \((w_{n(k)})_{k \in \mathbb{N}}\) and a sequence \((z_k)_{k \in \mathbb{N}}\) such that

(a) \( \lim_{k \to \infty} |x \in A \times I : z_k(x) \neq w_{n(k)}(x)| = 0; \)
(b) \( \left\{ \left( \partial_1 z_k \partial_2 z_k \frac{1}{p_{n(k)}} \partial_3 z_k \right) \right\} \) is equi-integrable;
(c) \( z_k \rightharpoonup \psi \) weakly in \( W^{1,p}(A \times I, \mathbb{R}^3) \).

The following proposition is the simple case of \([DM93, \text{Proposition 11.9.}]\).

**Proposition A.7.** Let \( X \) be a finite dimensional vector space over the real numbers and \( F : X \to [0, +\infty) \) an arbitrary function. If

a) \( F(0) = 0; \)

b) \( F(tx) \leq t^2 F(x) \) for every \( x \in X \) and for every \( t > 0; \)

c) \( F(x+y) + F(x-y) \leq 2F(x) + 2F(y) \) for every \( x,y \in X, \)

then \( F \) is a quadratic form. Conversely if \( F \) is a quadratic form then (a), (b), (c) are satisfied, and, in addition,

d) \( F(tx) = t^2 F(x) \) for every \( x \in X \) and for every \( t \in \mathbb{R} \) with \( t \neq 0; \)

e) \( F(x+y) + F(x-y) = 2F(x) + 2F(y) \), for every \( x,y \in X. \)

If \( \omega \) is a Lipschitz domain by \( A = A(\omega) \) we denote the class of all open subsets of \( \omega; \) while by \( B = B(\omega) \) we denote the class of all Borel subsets of \( \omega. \) By \( A_0 \) we denote the class of all open sets of \( \omega \) that are compactly contained in \( \omega. \) The following definitions and theorem can be found in \([DM93, \text{Chapter 14.}]\).

**Definition A.8.** For a function \( \alpha : A \to \mathbb{R} \) we say that it is increasing if \( \alpha(A) \leq \alpha(B) \), whenever \( A,B \in \mathcal{A}, \ A \subset B. \) We say that the increasing function \( \alpha : \mathcal{A} \to \mathbb{R} \) is inner regular if

\[
\alpha(A) = \sup \{ \alpha(B) : B \in \mathcal{A}, \ B \ll A \}. 
\]

**Definition A.9.** We say that a subset \( \mathcal{D} \) of \( \mathcal{A} \) is dense in \( \mathcal{A} \) if for every \( A,B \in \mathcal{A}, \) with \( A \ll B, \) there exists \( D \in \mathcal{D}, \) such that \( A \ll D \ll B. \)

**Remark 11.** If \( \alpha : \mathcal{A} \to \mathbb{R} \) is an increasing function and \( \mathcal{D} \) is the dense subset of \( \mathcal{A} \) then we have that

\[
\alpha(A) = \sup \{ \alpha(D) : D \in \mathcal{D}, \ D \ll A \}. 
\]

**Definition A.10.** Let \( \alpha : \mathcal{A} \to \mathbb{R} \) be non-negative increasing function. We say that

a) \( \alpha \) is subadditive on \( \mathcal{A} \) if \( \alpha(A) \leq \alpha(A_1) + \alpha(A_2) \) for every \( A,A_1,A_2 \in \mathcal{A} \) with \( A \subset A_1 \cup A_2; \)

b) \( \alpha \) is superadditive on \( \mathcal{A} \) if \( \alpha(A) \geq \alpha(A_1) + \alpha(A_2) \) for every \( A,A_1,A_2 \in \mathcal{A} \) with \( A_1 \cup A_2 \subset \mathcal{A} \) and \( A_1 \cap A_2 = \emptyset; \)

c) \( \alpha \) is a measure on \( \mathcal{A} \) if there exists a Borel measure \( \mu : B \to [0, +\infty] \) such that

\[
\alpha(A) = \mu(A) \text{ for every } A \in \mathcal{A}. 
\]

The following is \([DM93, \text{Theorem 14.23.}]\).

**Theorem A.11.** Let \( \alpha : \mathcal{A} \to [0, +\infty] \) be a non-negative increasing function such that \( \alpha(\emptyset) = 0. \) The following conditions are equivalent.

(i) \( \alpha \) is a measure on \( \mathcal{A}; \)

(ii) \( \alpha \) is subadditive, superadditive and inner regular on \( \mathcal{A}. \)
Remark 12. It can be seen that the measure $\mu$ which extends $\alpha$ is given by
$$\mu(E) = \inf \{ \alpha(A) : A \in \mathcal{A}, \ E \subset A \}.$$ 

We give a simple lemma about the sets with Lipschitz boundary.

**Lemma A.12.** Let $A \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. Then $A$ has finite number of connected components.

**Proof.** Denote by $\{ \Gamma_\alpha \}_{\alpha \in \Lambda}$ the connected components of $\partial A$. We want to prove that there is only finitely many such components. Suppose that there is infinite many such components. Then for each $n \in \mathbb{N}$ we can find $x_n \in \Gamma_n$, where $\Gamma_i \neq \Gamma_j$, for all $i \neq j$. Since $\partial A$ is compact we have that at least on a subsequence, still denoted by $(x_n)_{n \in \mathbb{N}}$, $x_n \rightarrow x \in \partial A$. Since $A$ has Lipschitz boundary we can find a Lipschitz frame around point $x \in \partial A$, with radius $\varepsilon > 0$. This means that there exists a bijective map $f_x : B(x, \varepsilon) \rightarrow B(0, 1)$ such that $f_x$ and $f_x^{-1}$ are Lipschitz continuous and such that $f_x(\partial A \cap B(x, \varepsilon)) = B(0, 1) \cap \{ x_n = 0 \}$ and $A \cap B(x, \varepsilon) = B(0, 1) \cap \{ x_n > 0 \}$. This contradicts the fact that $x_n \rightarrow x$ and that $x_n$ belong to different connected components of $\partial A$. Thus, we have proved that $\partial A$ has finitely many connected components. Take now all the connected components of the set $A$ and denote them by $\{ A_\alpha \}_{\alpha \in \Lambda}$. Using that $A$ has Lipschitz boundary it is easy to see that $\partial A \subset \bigcup_{\alpha \in \Lambda} \partial A_\alpha$. Then it is easy to check that $\partial A = \bigcup_{\alpha \in \Lambda} \partial A_\alpha$. Moreover it is easy to see that for $\alpha \neq \beta$, $\partial A_\alpha \cap \partial A_\beta = \emptyset$. Namely, if there is $x \in \partial A_\alpha \cap \partial A_\beta$, then by taking Lipschitz frame around $x$, it can be seen that $A_\alpha$ and $A_\beta$ would be connected. Also it is easy to see that every connected component of the boundary can be part of the boundary at most one of the connected component of $A$. This implies that there can be only finitely many connected components of $A$ and that they have disjoint closure. □

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**References**


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