Convergence of equilibria of thin inextensible von Kármán rods

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We study the effects of simultaneous homogenization and dimension reduction in the context of convergence of stationary points for thin inextensible non-homogeneous rods under the assumption of the von Karman scaling. Given $\Omega = (0, L) \times \omega \subset \mathbb{R}^3$ a three-dimensional rod-like canonical domain of length $L > 0$ and cross-section $\omega \subset \mathbb{R}^2$, the (scaled) energy functional of a rod of thickness $h > 0$ associated to a deformation $y^h : \Omega \to \mathbb{R}^3$ is defined by

$$\mathcal{E}^h(y^h) = \int_\Omega W^h(x, \nabla_h y^h)\mathrm{d}x - \int_\Omega f^h \cdot y^h \mathrm{d}x\,. \tag{1}$$

Here W^h is the elastic energy density describing a composite material, $\nabla_h y^h = (\partial_1 y^h \mid \frac{1}{h}\partial_2 y^h \mid \frac{1}{h}\partial_3 y^h)$ denotes the scaled gradient of the deformation, and f^h describes an external load. Different scalings with respect to the thickness h in the applied load and elastic energy lead at the limit to different rod models [3, 5]. We consider the *von Kármán scaling*, i.e. a minimizing sequence (y^h) satisfies

$$\limsup_{h\downarrow 0} \frac{1}{h^4}\int_\Omega W^h(x, \nabla_h y^h)\mathrm{d}x < \infty\,, \tag{2}$$

while the forcing term scales as $f^h = h^3 f$ with $f \in L^2((0, L), \mathbb{R}^3)$. Under assumption (2) one can prove, using the rigidity estimate [2], that minimizers y^h of $\mathcal{E}^h$ are close to a rigid body motion and its rigid transformations $\hat{y}^h$ converge to the identity on $(0, L)$ in the L^2-norm [4]. Furthermore, sequences of scaled displacements (u^h), (v_2^h), (v_3^h) and twist functions (w^h), defined by:

$$u^h(x_1) = \int_\omega \frac{\hat{y}_1^h - x_1}{h^2}\mathrm{d}x'\,, \quad v_i^h(x_1) = \int_\omega \frac{\hat{y}_i^h}{h}\mathrm{d}x'\,,\ \ i = 2, 3\,, \quad w^h(x_1) = \int_\omega \frac{x_2\hat{y}_3^h - x_3\hat{y}_2^h}{h^2}\mathrm{d}x'$$

converge weakly (on a subsequence) to $(u, v_2, v_3, w) \in H^1(0, L) \times H^2(0, L) \times H^2(0, L) \times H^1(0, L)$. Our work asserts that if $\hat{y}^h$ are also stationary points of $\mathcal{E}^h$ (in addition to being minimizers), i.e. $\hat{y}^h$ solves the Euler–Lagrange equation

$$\int_\Omega \Big(DW^h(x, \nabla_h y^h) : \nabla_h \phi - h^3(f_2\phi_2 + f_3\phi_3)\Big)\,\mathrm{d}x = 0\,, \tag{3}$$

for all test functions $\phi \in H^1_\omega(\Omega, \mathbb{R}^3) = \{\phi \in H^1(\Omega)\ :\ \phi|_{\{0\}\times\omega} = 0\}$, then (u, v_2, v_3, w) is a stationary point of the functional $\mathcal{E}^0$, which is the Γ-limit of the sequence of functionals $(\mathcal{E}^h)$ [1].

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