EQUATIONS FOR COEFFICIENTS OF TACTICAL DECOMPOSITION MATRICES FOR t-DESIGNS

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Abstract. Equations for coefficients of tactical decomposition matrices for 2-designs are well-known and they have been used for constructions of many examples of 2-designs. In this paper, we generalize these equations and propose an explicit equation system for coefficients of tactical decomposition matrices for t-(v, k, λ_t) designs, for any integer value of t.

1. Introduction and preliminary results

In this paper we give a contribution to the study of tactical decompositions of t-designs. The main result is given in Theorem 2.4 where we describe an equation system for the coefficients of tactical decomposition matrices for t-(v, k, λ_t) designs.

The idea of considering a tactical decomposition of a 2-design was first introduced by Dembowski [1]. Equations for coefficients when $t = 2$ are well-known [2] and they were used for constructions of many examples of 2-designs (examples are listed in [6]). The tactical decomposition was here induced by the action of a proposed automorphism group. In many cases it was enough to assume an action of a cyclic group of automorphisms, while in some larger ones, semidirect products of cyclic groups of prime order proved to be a feasible additional assumption.

In [4] we introduced how a tactical decomposition of a t-design induced by the action of a proposed automorphism group can be used for the enhancement of the Kramer-Mesner method [3]. An extensive usage of Kramer-Mesner method for t-designs can be found in [5]. Here, automorphism groups were mostly chosen between (projective) linear groups and (projective) special linear groups. Since the number of blocks of t-designs for larger t becomes soon quite large, it was necessary to assume actions of much larger automorphism groups.

Let t, v, k, λ_t be positive integers, such that $v > k \geq t$. A t-(v, k, λ_t) design is a finite incidence structure $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v elements called *points*, and $\mathcal{B}$ is a multiset of nonempty k-subsets of $\mathcal{P}$ called *blocks* such that every set of t distinct points is contained in exactly λ_t blocks. It is known that every t-(v,k,λ_t) design is also an s-(v,k,λ_s) design, $0 \le s < t$, where $\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}$. Therefore, each point is contained in $r = \lambda_1 = \lambda_t \binom{v-1}{t-1} / \binom{k-1}{t-1}$ blocks, and the number of blocks b equals to $\lambda_0 = \lambda_t \binom{v}{t} / \binom{k}{t}$.

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$ be the point set and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ the block set of a t-(v,k,λ_t)-design. The *incidence matrix* of a design is a 0-1 matrix $M = [m_{ij}]$ of type $v \times b$, where m_{ij} denotes the incidence of the point p_i and the block B_j.

A *decomposition* of a design is any partition $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m$ of the point set and $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ of the block set. We say that a decomposition is *tactical* if there are nonnegative integers ρ_{ij} and κ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, such that each point of $\mathcal{P}_i$ lies in exactly ρ_{ij} blocks of $\mathcal{B}_j$, and each block of $\mathcal{B}_j$ contains exactly κ_{ij} points from $\mathcal{P}_i$. Matrices $[\rho_{ij}]$ and $[\kappa_{ij}]$ are called *tactical decomposition matrices*.

An *automorphism* of a design $(\mathcal{P}, \mathcal{B})$ is a mapping $\pi : \mathcal{P} \to \mathcal{P}$ such that if $B \in \mathcal{B}$, then $\pi(B) \in \mathcal{B}$. The set of all automorphisms of a design $(\mathcal{P}, \mathcal{B})$ is a group and any of its subgroups partitions the point set $\mathcal{P}$ and the block set $\mathcal{B}$ of a design into orbits. It is well known that such a decomposition is tactical.

Assume now that a design with parameters t-(v,k,λ_t) has a tactical decomposition. For $p \in \mathcal{P}$ we denote by $\langle p \rangle = \{B \in \mathcal{B} \mid p \in B\}$. Then

$$\begin{aligned} \rho_{ij} &= |\langle p \rangle \cap \mathcal{B}_j|, \;\; p \in \mathcal{P}_i, \\ \kappa_{ij} &= |B \cap \mathcal{P}_i|, \;\; B \in \mathcal{B}_j. \end{aligned}$$

These numbers do not depend on the choice of $p \in \mathcal{P}_i$ and $B \in \mathcal{B}_j$ if and only if the decomposition is tactical. Obviously holds

$$\sum_{i=1}^{m} \kappa_{ij} = k, \;\; \sum_{j=1}^{n} \rho_{ij} = \lambda_1. \tag{1}$$

Coefficients of tactical decomposition matrices for a 2-(v,k,λ_2) design satisfy following equations:

$$|\mathcal{P}_i| \cdot \rho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij}\,. \tag{2}$$

$$\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_2 j} = \begin{cases} \lambda_2 \cdot |\mathcal{P}_{i_2}|, & \text{for } i_1 \neq i_2, \\ \lambda_1 + \lambda_2 \cdot (|\mathcal{P}_{i_1}| - 1), & \text{for } i_1 = i_2. \end{cases} \tag{3}$$

The latter of the equations can be obtained in the following way. Fixing a point $p \in \mathcal{P}_{i_1}$ and double counting of the set

$$\{(q, B) \mid q \in \mathcal{P}_{i_2}, p \in B, q \in B\}$$

yields

$$\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_2 j} = \sum_{q \in \mathcal{P}_{i_2}} |\langle p \rangle \cap \langle q \rangle|. \tag{4}$$

Computing the right-hand side, we get (3).

2. Tactical decompositions of t-designs

In this section we show how equations (3) for 2-(v, k, λ_2) designs can be generalized into an equation system for the coefficients of tactical decomposition matrices for t-(v, k, λ_t) designs, $t \in \mathbb{N}$. Solutions of such an equation system represent necessary conditions for the existence of t-designs and can be further used either to implement the tactical decomposition method by blowing up the condensed matrices to full incidence matrices, or to reduce the number of columns in the Kramer-Mesner matrix and the size of the integer linear system which needs to be solved.

Fix an integer $0 < s < t$ and a point $p \in \mathcal{P}_{i_1}$, and consider the set

$$\{(q_1, \ldots, q_s, B) \mid q_1 \in \mathcal{P}_{i_1}, \ldots, q_s \in \mathcal{P}_{i_s}, p \in B, q_1, \ldots, q_s \in B\}.$$

Double counting of this set gives a formula analogous to (4)

$$\begin{aligned} &\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_2 j} \cdots \kappa_{i_s j} = \\ &\qquad \sum_{q_1 \in \mathcal{P}_{i_1}} \sum_{q_2 \in \mathcal{P}_{i_2}} \cdots \sum_{q_s \in \mathcal{P}_{i_s}} |\langle p \rangle \cap \langle q_1 \rangle \cap \cdots \cap \langle q_s \rangle|. \end{aligned} \tag{5}$$

For more details, the reader is referred to [4]. Note that in [4] no explicit expression for the right-hand side of the previous formula was given. In order to compute it, we need two preparatory lemmas. The number of partitions of an n-set into k nonempty subsets shall be denoted by $\left\{ {n \atop k} \right\}$ (these are the Stirling numbers of second kind), and the falling factorial by $(x)_n = x(x-1)\cdots(x-n+1)$.

Lemma 2.1. *Let $1 \le s \le t$. Then*

$$\sum_{(q_1, \ldots, q_s) \in \mathcal{P}_i^s} |\langle q_1 \rangle \cap \ldots \cap \langle q_s \rangle| = \sum_{\omega=1}^{s} \lambda_\omega \left\{ {s \atop \omega} \right\} (|\mathcal{P}_i|)_\omega. \tag{6}$$

Proof. For a fixed $q = (q_1, \dots, q_s) \in \mathcal{P}_i^s$, we denote by $Q = \{q_1, \dots, q_s\}$ the set of all distinct points in q. If $|Q| = \omega$, then the number of blocks containing all points of Q is λ_ω, that is

$$|\langle q_1 \rangle \cap \ldots \cap \langle q_s \rangle| = \lambda_\omega.$$

On the other hand, for a fixed ω

$$|\{q \in \mathcal{P}_i^s \,:\, |Q| = \omega\}| = \left\{ {s \atop \omega} \right\} (|\mathcal{P}_i|)_\omega.$$

The lemma now follows from

$$\{q \in \mathcal{P}_i^s\} = \bigsqcup_{\omega=1}^{s} \{q \in \mathcal{P}_i^s \,:\, |Q| = \omega\}.$$

□ □

Lemma 2.2. *Let $1 \le s \le t$ and let $p \in \mathcal{P}_i$. Then*

$$\sum_{(q_2,\dots,q_s)\in\mathcal{P}_i^{s-1}} |\langle p \rangle \cap \langle q_2 \rangle \cap \ldots \cap \langle q_s \rangle| = \sum_{\omega=1}^{s} \lambda_\omega \left\{ {s \atop \omega} \right\} (|\mathcal{P}_i| - 1)_{\omega-1}. \tag{7}$$

Proof. As in the previous lemma, for a fixed $q = (q_2, \dots, q_s) \in \mathcal{P}_i^{s-1}$, we denote by $Q_p = \{p, q_2, \dots, q_s\}$. If $|Q_p| = \omega$, then

$$|\langle p \rangle \cap \langle q_2 \rangle \cap \ldots \cap \langle q_s \rangle| = \lambda_\omega.$$

On the other hand, for a fixed ω

$$|\{q \in \mathcal{P}_i^{s-1} \,:\, |Q_p| = \omega\}| = \left\{ {s \atop \omega} \right\} (|\mathcal{P}_i| - 1)_{\omega-1}.$$

The lemma is now an immediate consequence of

$$\{q \in \mathcal{P}_i^{s-1}\} = \bigsqcup_{\omega=1}^{s} \{q \in \mathcal{P}_i^{s-1} \,:\, |Q_p| = \omega\}.$$

□ □

Lemma 2.3. *Let $1 \le s \le t$, and let $m_1, \dots, m_s$ be positive integers, such that $m_1 + \cdots + m_s \le t$. Let $\mathcal{P}_{i_1}, \dots, \mathcal{P}_{i_s}$ be mutually distinct and let $p \in \mathcal{P}_{i_1}$. Then*

$$\begin{aligned} &\sum_{q\in\mathcal{Q}} |\langle p \rangle \cap \langle q_2^1 \rangle \cap \ldots \cap \langle q_{m_1}^1 \rangle \cap \ldots \cap \langle q_1^s \rangle \cap \ldots \cap \langle q_{m_s}^s \rangle| = \\ &\sum_{\omega\in\Omega} \lambda_{\omega_1+\omega_2+\cdots+\omega_s} \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1-1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}, \end{aligned} \tag{8}$$

where

$$\mathcal{Q} = \{(q_2^1, \dots, q_{m_1}^1, q_1^2, \dots, q_{m_s}^2, \dots, q_1^s, \dots, q_{m_s}^s)$$

$$\in \mathcal{P}_{i_1}^{m_1-1} \times \mathcal{P}_{i_2}^{m_2} \times \cdots \times \mathcal{P}_{i_s}^{m_s}\}$$

and

$$\Omega = \{(\omega_1, \ldots, \omega_s) \,:\, 1 \leq \omega_j \leq m_j\}.$$

Proof. For a fixed $q \in \mathcal{Q}$ we denote now

$$Q_1 = \{p, q_2^1, \ldots, q_{m_1}^1\}, \;\; |Q_1| = \omega_1,$$

and

$$Q_j = \{q_1^j, \ldots, q_{m_j}^j\}, \;\; |Q_j| = \omega_j, \;\; 2 \leq j \leq s.$$

It is easy to see that then

$$|\langle p\rangle \cap \langle q_2^1\rangle \cap \cdots \cap \langle q_{m_s}^s\rangle| = \lambda_{\omega_1+\cdots+\omega_s}.$$

On the other hand, following from the previous lemmas, for a fixed $(\omega_1, \ldots, \omega_s)$, we get

$$|\{q \in \mathcal{Q} \,:\, |Q_j| = \omega_j\}| = \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1 - 1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}.$$

The lemma follows from

$$\{q \in \mathcal{Q}\} = \bigsqcup_{\omega \in \Omega} \{q \in \mathcal{Q} \,:\, |Q_j| = \omega_j, 1 \leq j \leq s\}.$$

□ □

Applying the result from the previous lemma to the equality (5), we obtain an equation system for the coefficients of tactical decomposition matrices for a t-(v, k, λ_t) design.

Theorem 2.4. *Assume $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ_t) design with a tactical decomposition $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m$ and $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$. Let $[\rho_{ij}]$ and $[\kappa_{ij}]$ be corresponding tactical decomposition matrices. Let $1 \leq s \leq t$ and let $m_1, \ldots, m_s$ be positive integers, such that $m_1 + \cdots + m_s \leq t$. Let $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_s}$ be mutually distinct. Then*

$$\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_1 j}^{m_1 - 1} \kappa_{i_2 j}^{m_2} \cdots \kappa_{i_s j}^{m_s} = \sum_{\omega \in \Omega} \lambda_{\omega_1 + \omega_2 + \cdots + \omega_s} \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1 - 1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}, \tag{9}$$

where

$$\Omega = \{(\omega_1, \ldots, \omega_s) \,:\, 1 \leq \omega_j \leq m_j\}.$$

The right-hand side of (9) can be expressed in a form more convenient for computation:

$$\sum_{\omega_1=1}^{m_1}\sum_{\omega_2=1}^{m_2}\cdots\sum_{\omega_s=1}^{m_s}\lambda_{\omega_1+\cdots+\omega_s}\left\{{m_1 \atop \omega_1}\right\}(|\mathcal{P}_{i_1}|-1)_{\omega_1-1}\prod_{j=2}^{s}\left\{{m_j \atop \omega_j}\right\}(|\mathcal{P}_{i_j}|)_{\omega_j}.$$

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