Osnovne trigonometrijske jednakosti

$$\sin\Big(x \pm \frac{\pi}{2}\Big) = \pm\cos x$$
$$\cos\Big(x \pm \frac{\pi}{2}\Big) = \mp\sin x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$
$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$
$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$2\sin^2 x = 1 - \cos(2x)$$
$$2\cos^2 x = 1 + \cos(2x)$$

Tablice suma i integrala

Konačne sume

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$
$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$
$$\sum_{i=0}^{n} e^{j(\theta+i\phi)} = \frac{\sin\big((n+1)\phi/2\big)}{\sin(\phi/2)} e^{j(\theta+n\phi/2)}$$
$$\sum_{i=0}^{n} \binom{n}{i} = \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} = 2^n$$

Neodređeni integrali

Racionalne funkcije

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}, \quad 0 < n$$
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$$
$$\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\,\mathrm{tg}^{-1}\Big(\frac{ax}{b}\Big)$$
$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$$
$$\int \frac{x^2\,dx}{a^2+x^2} = x - a\,\mathrm{tg}^{-1}\Big(\frac{x}{a}\Big)$$
$$\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\,\mathrm{tg}^{-1}\Big(\frac{x}{a}\Big)$$
$$\int \frac{x\,dx}{(a^2+x^2)^2} = \frac{-1}{2(a^2+x^2)}$$
$$\int \frac{x^2\,dx}{(a^2+x^2)^2} = \frac{-x}{2(a^2+x^2)} + \frac{1}{2a}\,\mathrm{tg}^{-1}\Big(\frac{x}{a}\Big)$$

Trigonometrijske funkcije

$$\int \cos(x)\,dx = \sin(x)$$
$$\int x\cos(x)\,dx = \cos(x) + x\sin(x)$$
$$\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2-2)\sin(x)$$
$$\int \sin(x)\,dx = -\cos(x)$$
$$\int x\sin(x)\,dx = \sin(x) - x\cos(x)$$
$$\int x^2\sin(x)\,dx = 2x\sin(x) + (2-x^2)\cos(x)$$

Eksponencijalne funkcije

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$
$$\int xe^{ax}\,dx = \Big(\frac{x}{a} - \frac{1}{a^2}\Big)e^{ax}$$
$$\int x^2e^{ax}\,dx = \Big(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\Big)e^{ax}$$
$$\int x^3e^{ax}\,dx = \Big(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\Big)e^{ax}$$
$$\int \sin(x)e^{ax}\,dx = \frac{1}{a^2+1}\big(a\sin(x) - \cos(x)\big)e^{ax}$$
$$\int \cos(x)e^{ax}\,dx = \frac{1}{a^2+1}\big(a\cos(x) + \sin(x)\big)e^{ax}$$

Određeni integrali

$$\int_{-\infty}^{+\infty} e^{-a^2x^2+bx}\,dx = \frac{\sqrt{\pi}}{a}e^{\frac{b^2}{4a^2}}, \quad a>0$$

$$\int_{0}^{+\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$$

$$\int_{0}^{+\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_{0}^{+\infty} \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}$$

Laplaceova transformacija

Jednostrana Laplaceova transformacija funkcije $x(t)$ je:

$$\mathcal{L}\big(x(t)\big) = \int_{0^-}^{+\infty} x(t)e^{-st}dt = X(s)$$

Kažemo da su $x(t)$ i $X(s)$ transformacijski par i pišemo $x(t) \circ\!\!-\!\!\bullet X(s)$.

Tablica transformacija

$\delta(t) \circ\!\!-\!\!\bullet 1$

$\delta(t-t_0) \circ\!\!-\!\!\bullet e^{-st_0}, \quad t_0>0$

$\mu(t) \circ\!\!-\!\!\bullet \frac{1}{s}$

$t\,\mu(t) \circ\!\!-\!\!\bullet \frac{1}{s^2}$

$e^{-at}\,\mu(t) \circ\!\!-\!\!\bullet \frac{1}{s+a}$

$te^{-at}\,\mu(t) \circ\!\!-\!\!\bullet \frac{1}{(s+a)^2}$

$\cos(\omega_0 t)\,\mu(t) \circ\!\!-\!\!\bullet \frac{s}{s^2+\omega_0^2}$

$\sin(\omega_0 t)\,\mu(t) \circ\!\!-\!\!\bullet \frac{\omega_0}{s^2+\omega_0^2}$

$\frac{1}{b-a}(e^{-at}-e^{-bt})\,\mu(t) \circ\!\!-\!\!\bullet \frac{1}{(s+a)(s+b)}$

$\frac{1}{a-b}(ae^{-at}-be^{-bt})\,\mu(t) \circ\!\!-\!\!\bullet \frac{s}{(s+a)(s+b)}$

$\frac{1}{a}e^{-bt}\sin(at)\,\mu(t) \circ\!\!-\!\!\bullet \frac{1}{(s+b)^2+a^2}$

$e^{-bt}\big(\cos(at)-\frac{b}{a}\sin(at)\big)\,\mu(t) \circ\!\!-\!\!\bullet \frac{s}{(s+b)^2+a^2}$

Vremenski kontinuirana Fourierova transformacija

Vremenski kontinuirana Fourierova transformacija (CTFT – *Continuous-Time Fourier Transform*) funkcije $x(t)$ je:

$$\mathrm{CTFT}\big(x(t)\big) = X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

Inverzna transformacija je:

$$\mathrm{ICTFT}\big(X(j\omega)\big) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Kažemo da su $x(t)$ i $X(j\omega)$ transformacijski par i pišemo $x(t) \circ\!\!-\!\!\bullet X(j\omega)$.

Dovoljni (ali ne i nužni) uvjeti za postojanje transformacije funkcije $x(t)$ su:

1. Funkcija x(t) zadovoljava Dirichletove uvjete (funkcija je ograničena s konačnim brojem maksimuma i minimuma te konačnim brojem diskontinuiteta u bilo kojem konačnom vremenskom intervalu).
2. $\int_{-\infty}^{+\infty} |x(t)|dt < \infty$

Svojstva Fourierove transformacije

Neka je $x(t) \circ\!\!-\!\!\bullet X(j\omega)$ i neka su α_i, t_0 i ω_0 konstante. Fourierova transformacija tada zadovoljava sljedeća svojstva:

Linearnost

$$x(t) = \sum_{i=1}^{n}\alpha_i x_i(t) \circ\!\!-\!\!\bullet \sum_{i=1}^{n}\alpha_i X_i(j\omega) = X(j\omega)$$

Dualnost

$$X(t) \circ\!\!-\!\!\bullet 2\pi x(-\omega)$$

Pomak u vremenu i frekvenciji

$$x(t-t_0) \circ\!\!-\!\!\bullet X(j\omega)e^{-j\omega t_0}$$

$$x(t)e^{j\omega_0 t} \circ\!\!-\!\!\bullet X(j\omega - j\omega_0)$$

Skaliranje

$$x(\alpha t) \circ\!\!-\!\!\bullet \frac{1}{|\alpha|}X\Big(\frac{j\omega}{\alpha}\Big)$$

Deriviranje

$$\frac{d^n x(t)}{dt^n} \circ\!\!-\!\!\bullet \; (j\omega)^n X(j\omega)$$

$$(-jt)^n x(t) \circ\!\!-\!\!\bullet \; \frac{d^n X(j\omega)}{d\omega^n}$$

Integriranje

$$\int_{-\infty}^{t} x(\tau)\, d\tau \circ\!\!-\!\!\bullet \; \pi X(0)\, \delta(\omega) + \frac{X(j\omega)}{j\omega}$$

$$\pi x(0)\, \delta(t) - \frac{x(t)}{jt} \circ\!\!-\!\!\bullet \; \int_{-\infty}^{\omega} X(j\xi)\, d\xi$$

Konjugacija

$$x^*(t) \circ\!\!-\!\!\bullet \; X^*(-j\omega)$$

$$x^*(-t) \circ\!\!-\!\!\bullet \; X^*(j\omega)$$

Konvolucija

$$\int_{-\infty}^{+\infty} x_1(\tau) x_2(t-\tau)\, d\tau \circ\!\!-\!\!\bullet \; X_1(j\omega) X_2(j\omega)$$

$$x_1(t) x_2(t) \circ\!\!-\!\!\bullet \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1(j\xi) X_2(j\omega - j\xi)\, d\xi$$

Korelacija

$$\int_{-\infty}^{+\infty} x_1^*(\tau) x_2(t+\tau)\, d\tau \circ\!\!-\!\!\bullet \; X_1^*(j\omega) X_2(j\omega)$$

$$x_1^*(t) x_2(t) \circ\!\!-\!\!\bullet \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1^*(j\xi) X_2(j\omega + j\xi)\, d\xi$$

Parsevalov teorem

$$\int_{-\infty}^{+\infty} x_1^*(t) x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1^*(j\omega) X_2(j\omega)\, d\omega$$

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

Tablica transformacija

Neka je:

$$\mu(x) = \begin{cases} 1, \; x > 0 \\ 0, \; x < 0 \end{cases}$$

$$\mathrm{rect}(x) = \begin{cases} 1, & -\frac{1}{2} < \; x \; < \frac{1}{2} \\ 0, & \frac{1}{2} < |x| \end{cases}$$

$$\mathrm{tri}(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Uz te oznake važnije transformacije su:

$$1 \circ\!\!-\!\!\bullet \; 2\pi\, \delta(\omega)$$

$$\delta(t) \circ\!\!-\!\!\bullet \; 1$$

$$\mu(t) \circ\!\!-\!\!\bullet \; \pi\, \delta(\omega) + \frac{1}{j\omega}$$

$$\frac{1}{2}\, \delta(t) - \frac{1}{2\pi jt} \circ\!\!-\!\!\bullet \; \mu(\omega)$$

$$\mathrm{sgn}(t) \circ\!\!-\!\!\bullet \; \frac{2}{j\omega}$$

$$\mathrm{rect}\Big(\frac{t}{T}\Big) \circ\!\!-\!\!\bullet \; T\, \mathrm{sinc}\Big(\frac{\omega T}{2\pi}\Big)$$

$$\mathrm{sinc}(at) \circ\!\!-\!\!\bullet \; \frac{1}{a}\, \mathrm{rect}\Big(\frac{\omega}{2\pi a}\Big)$$

$$\mathrm{tri}\Big(\frac{t}{T}\Big) \circ\!\!-\!\!\bullet \; T\, \mathrm{sinc}^2\Big(\frac{\omega T}{2\pi}\Big)$$

$$\mathrm{sinc}^2(at) \circ\!\!-\!\!\bullet \; \frac{1}{a}\, \mathrm{tri}\Big(\frac{\omega}{2\pi a}\Big)$$

$$e^{j\omega_0 t} \circ\!\!-\!\!\bullet \; 2\pi\, \delta(\omega - \omega_0)$$

$$\delta(t - t_0) \circ\!\!-\!\!\bullet \; e^{-j\omega t_0}$$

$$\sin(\omega_0 t) \circ\!\!-\!\!\bullet \; -j\pi\big(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\big)$$

$$\cos(\omega_0 t) \circ\!\!-\!\!\bullet \; \pi\big(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big)$$

$$\sum_{i=-\infty}^{+\infty} \delta(t - iT_0) \circ\!\!-\!\!\bullet \; \frac{2\pi}{T_0} \sum_{i=-\infty}^{+\infty} \delta\Big(\omega - i\frac{2\pi}{T_0}\Big)$$

$$\sin(\omega_0 t)\, \mu(t) \circ\!\!-\!\!\bullet \; -\frac{j\pi}{2}\big(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\big) + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

$$\cos(\omega_0 t)\, \mu(t) \circ\!\!-\!\!\bullet \; \frac{\pi}{2}\big(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big) + \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$e^{-at}\, \mu(t) \circ\!\!-\!\!\bullet \; \frac{1}{a + j\omega}, \quad a > 0$$

$$te^{-at}\mu(t) \circ\!\!-\!\!\bullet \frac{1}{(a+j\omega)^2}, \quad a>0$$

$$t^2e^{-at}\mu(t) \circ\!\!-\!\!\bullet \frac{2}{(a+j\omega)^3}, \quad a>0$$

$$t^3e^{-at}\mu(t) \circ\!\!-\!\!\bullet \frac{6}{(a+j\omega)^4}, \quad a>0$$

$$e^{-a|t|} \circ\!\!-\!\!\bullet \frac{2a}{a^2+\omega^2}$$

$$e^{-\frac{t^2}{2a^2}} \circ\!\!-\!\!\bullet a\sqrt{2\pi}e^{-a^2\omega^2/2}$$

Vremenski kontinuiran Fourierov red

Vremenski kontinuiran Fourierov red (CTFS – *Continuous-Time Fourier Series*) periodične funkcije $x(t)$ s periodom T_0 je:

$$\mathrm{CTFS}_{T_0}\big(x(t)\big) = X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

Inverzna transformacija je:

$$\mathrm{ICTFS}_{T_0}\big(X_k\big) = x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{j\omega_0 kt}$$

Pri tome je $\omega_0 = \frac{2\pi}{T_0}$. Kažemo da su $x(t)$ i X_k transformacijski par i pišemo $x(t) \circ\!\!-\!\!\bullet X_k$.

Vremenski diskretna Fourierova transformacija

Vremenski diskretna Fourierova transformacija (DTFT – *Discrete-Time Fourier Transform*) niza $x(n)$ je:

$$\mathrm{DTFT}\big(x(n)\big) = X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\Omega n}$$

Inverzna transformacija je:

$$\mathrm{IDTFT}\big(X(e^{j\Omega})\big) = x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

Niz $x(n)$ i njegov spektar $X(e^{j\Omega})$ čine transformacijski par $x(n) \circ\!\!-\!\!\bullet X(e^{j\Omega})$.

Dovoljan (ali ne i nužni) uvjet za postojanje transformacije niza $x(n)$ je apsolutna sumabilnost:

$$\sum_{n=-\infty}^{+\infty} \big|x(n)\big| < \infty$$

Svojstva vremenski diskretne Fourierove transformacije

Neka je $x(n) \circ\!\!-\!\!\bullet X(e^{j\Omega})$ i neka su α_i, n_0 i Ω_0 konstante. Vremenski diskretna Fourierova transformacija tada zadovoljava sljedeća svojstva:

Linearnost

$$x(n) = \sum_{i=1}^{n} \alpha_i x_i(n) \circ\!\!-\!\!\bullet \sum_{i=1}^{n} \alpha_i X_i(e^{j\Omega}) = X(e^{j\Omega})$$

Pomak u vremenu i frekvenciji

$$x(n-n_0) \circ\!\!-\!\!\bullet X(e^{j\Omega})e^{-j\Omega n_0}$$

$$x(n)e^{j\Omega_0 n} \circ\!\!-\!\!\bullet X(e^{j\Omega - j\Omega_0})$$

Deriviranje i diferenciranje

$$\Delta x(n) \circ\!\!-\!\!\bullet (e^{j\Omega}-1)X(e^{j\Omega})$$

$$n^i x(n) \circ\!\!-\!\!\bullet j^i \frac{d^i X(e^{j\Omega})}{d\Omega^i}$$

Sumiranje

$$\sum_{i=-\infty}^{n} x(i) \circ\!\!-\!\!\bullet \frac{1}{1-e^{-j\Omega}}X(e^{j\Omega})$$

Konjugacija

$$x^*(n) \circ\!\!-\!\!\bullet X^*(e^{-j\Omega})$$

$$x^*(-n) \circ\!\!-\!\!\bullet X^*(e^{j\Omega})$$

Konvolucija

$$\sum_{i=-\infty}^{+\infty} x_1(i)x_2(n-i) \circ\!\!-\!\!\bullet X_1(e^{j\Omega})X_2(e^{j\Omega})$$

$$x_1(n)x_2(n) \circ\!\!-\!\!\bullet \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\xi})X_2(e^{j\Omega-j\xi})\,d\xi$$

Parsevalov teorem

$$\sum_{n=-\infty}^{+\infty} x_1^*(n)x_2(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1^*(e^{j\Omega})X_2(e^{j\Omega})\,d\Omega$$

$$\sum_{n=-\infty}^{+\infty} \big|x(n)\big|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \big|X(e^{j\Omega})\big|^2\,d\Omega$$

Relacije simetričnosti

Neka je $x(n)$ čisto realan niz i neka je $x(n) \circ\!\!-\!\!\bullet X(e^{j\Omega})$. Tada je:

$$\frac{1}{2}\big(x(n) + x(-n)\big) \circ\!\!-\!\!\bullet \operatorname{Re}\big(X(e^{j\Omega})\big)$$

$$\frac{1}{2}\big(x(n) - x(-n)\big) \circ\!\!-\!\!\bullet j\operatorname{Im}\big(X(e^{j\Omega})\big)$$

Također vrijedi:

$$X(e^{j\Omega}) = X^*(e^{-j\Omega})$$

$$\operatorname{Re}\big(X(e^{j\Omega})\big) = \operatorname{Re}\big(X(e^{-j\Omega})\big)$$

$$\operatorname{Im}\big(X(e^{-j\Omega})\big) = -\operatorname{Im}\big(X(e^{j\Omega})\big)$$

Tablica transformacija

$$\delta(n) \circ\!\!-\!\!\bullet 1$$

$$1 \circ\!\!-\!\!\bullet \sum_{i=-\infty}^{+\infty} 2\pi\delta(\Omega + 2\pi i)$$

$$e^{j\Omega_0 n} \circ\!\!-\!\!\bullet \sum_{i=-\infty}^{+\infty} 2\pi\delta(\Omega - \Omega_0 + 2\pi i)$$

$$\mu(n) \circ\!\!-\!\!\bullet \frac{1}{1 - e^{-j\Omega}} + \sum_{i=-\infty}^{+\infty} \pi\delta(\Omega + 2\pi i)$$

$$a^n\,\mu(n) \circ\!\!-\!\!\bullet \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

$$na^n\,\mu(n) \circ\!\!-\!\!\bullet \frac{ae^{-j\Omega}}{(1 - ae^{-j\Omega})^2}, \quad |a| < 1$$

$\sin(\Omega_0 n)$

$$\circ\!\!-\!\!\bullet \sum_{i=-\infty}^{+\infty} j\pi\big(\delta(\Omega + \Omega_0 + 2\pi i) - \delta(\Omega - \Omega_0 + 2\pi i)\big)$$

$\cos(\Omega_0 n)$

$$\circ\!\!-\!\!\bullet \sum_{i=-\infty}^{+\infty} \pi\big(\delta(\Omega + \Omega_0 + 2\pi i) + \delta(\Omega - \Omega_0 + 2\pi i)\big)$$

$a^n \sin(\Omega_0 n)\,\mu(n)$

$$\circ\!\!-\!\!\bullet \frac{ae^{j\Omega}\sin(\Omega_0)}{e^{2j\Omega} - 2ae^{j\Omega}\cos(\Omega_0) + a^2}, \quad |a| < 1$$

$a^n \cos(\Omega_0 n)\,\mu(n)$

$$\circ\!\!-\!\!\bullet \frac{e^{j\Omega}\big(e^{j\Omega} - a\cos(\Omega_0)\big)}{e^{2j\Omega} - 2ae^{j\Omega}\cos(\Omega_0) + a^2}, \quad |a| < 1$$

Vremenski diskretan Fourierov red

Vremenski diskretan Fourierov red (DTFS – *Discrete-Time Fourier Series*) periodičnog niza $x(n)$ perioda N je:

$$\operatorname{DTFS}_N\big(x(n)\big) = X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-2\pi jkn/N}$$

Inverzna transformacija je:

$$\operatorname{IDTFS}_N\big(X_k\big) = x(n) = \sum_{k=0}^{N-1} X_k e^{2\pi jkn/N}$$

Niz $x(n)$ i njegov spektar X_k čine transformacijski par $x(n) \circ\!\!-\!\!\bullet X_k$.

Diskretna Fourierova transformacija

Diskretna Fourierova transformacija (DFT – *Discrete Fourier Transform*) konačnog niza $x(n)$ duljine N je:

$$\operatorname{DFT}_N\big(x(n)\big) = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}, \quad 0 \le k < N$$

Pri tome je $W_N^{nk} = e^{-2\pi jnk/N}$. Inverzna transformacija je:

$$\operatorname{IDFT}_N\big(X(k)\big) = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}, \quad 0 \le n < N$$

Niz $x(n)$ i njegov spektar $X(k)$ čine transformacijski par $x(n) \circ\!\!-\!\!\bullet X(k)$.

Svojstva diskretne Fourierove transformacije

Neka je $x(n) \circ\!\!-\!\!\bullet X(k)$ i neka su α_i, n_0 i k_0 konstante. DFT tada zadovoljava sljedeća svojstva:

Linearnost

$$x(n) = \sum_{i=1}^{n} \alpha_i x_i(n) \circ\!\!-\!\!\bullet \sum_{i=1}^{n} \alpha_i X_i(k) = X(k)$$

Dualnost

$$X(n) \circ\!\!-\!\!\bullet Nx\big(\langle -k\rangle_N\big)$$

Cirkularni pomak u vremenu i frekvenciji

$$x\big(\langle n - n_0\rangle_N\big) \circ\!\!-\!\!\bullet X(k)W_N^{kn_0}$$

$$x(n)W_N^{k_0 n} \circ\!\!-\!\!\bullet X\big(\langle k - k_0\rangle_N\big)$$

Cirkularna konvolucija

$$\sum_{i=0}^{N-1} x_1(i)x_2\big(\langle n-i\rangle_N\big) \circ\!\!-\!\!\bullet\; X_1(k)X_2(k)$$

$$x_1(n)x_2(n) \circ\!\!-\!\!\bullet\; \frac{1}{N}\sum_{i=0}^{N-1} X_1(i)X_2\big(\langle k-i\rangle_N\big)$$

Parsevalova relacija

$$\sum_{n=0}^{N-1} x_1^*(n)x_2(n) \circ\!\!-\!\!\bullet\; \frac{1}{N}\sum_{k=0}^{N-1} X_1^*(k)X_2(k)$$

$$\sum_{n=0}^{N-1} \big|x(n)\big|^2 \circ\!\!-\!\!\bullet\; \frac{1}{N}\sum_{k=0}^{N-1} \big|X(k)\big|^2$$

$\mathcal{Z}$-transformacija

$\mathcal{Z}$-transformacija niza $f(n)$ je: $\mathcal{Z}\big(f(n)\big) = \sum_{n=0}^{+\infty} f(n)z^{-n}$

Svojstva $\mathcal{Z}$ transformacije

Neka je $\mathcal{Z}\big(f(n)\big) = F(z)$ i $\mathcal{Z}\big(g(n)\big) = G(z)$. Tada vrijedi:

Linearnost

$$f(n) = \sum_{i=1}^{n} \alpha_i f_i(n) \circ\!\!-\!\!\bullet\; \sum_{i=1}^{n} \alpha_i F_i(z) = F(z)$$

Pomak

$$f(n+1) \circ\!\!-\!\!\bullet\; zF(z) - zf(0)$$

$$f(n+m) \circ\!\!-\!\!\bullet\; z^m F(z) - \sum_{i=0}^{m-1} f(i)z^{m-i}$$

$$f(n-1) \circ\!\!-\!\!\bullet\; \frac{1}{z}F(z) + f(-1)$$

$$f(n-m) \circ\!\!-\!\!\bullet\; z^{-m}F(z) + \sum_{i=0}^{m-1} f(i-m)z^{-i}$$

Skaliranje

$$a^n f(n) \circ\!\!-\!\!\bullet\; F(\frac{z}{a})$$

Diferenciranje i deriviranje

$$\Delta f(n) \circ\!\!-\!\!\bullet\; (z-1)F(z)$$

$$nf(n) \circ\!\!-\!\!\bullet\; -z\frac{dF(z)}{dz}$$

Konvolucija

$$\sum_{i=0}^{+\infty} f(i)g(n-i) \circ\!\!-\!\!\bullet\; F(z)G(z)$$

Tablica transformacija

$$\delta(n) \circ\!\!-\!\!\bullet\; 1$$

$$\delta(n-m) \circ\!\!-\!\!\bullet\; z^{-m}$$

$$n \circ\!\!-\!\!\bullet\; \frac{z}{(z-1)^2}$$

$$1^n \circ\!\!-\!\!\bullet\; \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$a^n \circ\!\!-\!\!\bullet\; \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$(n+1)a^n \circ\!\!-\!\!\bullet\; \frac{z^2}{(z-a)^2}$$

$$\frac{(n+1)(n+2)}{2!}a^n \circ\!\!-\!\!\bullet\; \frac{z^3}{(z-a)^3}$$

$$\frac{(n+1)(n+2)\dots(n+m-1)}{(m-1)!}a^n \circ\!\!-\!\!\bullet\; \frac{z^m}{(z-a)^m}$$

$$\frac{n(n-1)(n-2)\dots(n-m+1)}{m!}a^{n-m} \circ\!\!-\!\!\bullet\; \frac{z}{(z-a)^{m+1}}$$

$$a^n - \delta(n) \circ\!\!-\!\!\bullet\; \frac{a}{z-a}$$

$$\sin(an) \circ\!\!-\!\!\bullet\; \frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$$

$$\cos(an) \circ\!\!-\!\!\bullet\; \frac{z^2 - z\cos(a)}{z^2 - 2z\cos(a) + 1}$$

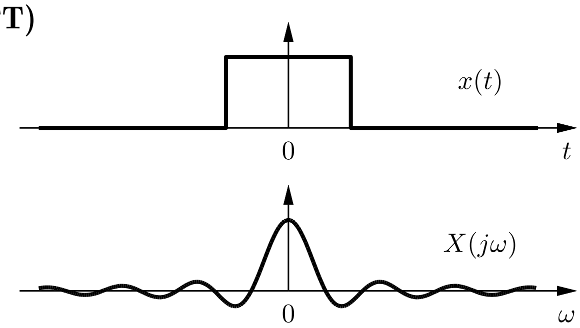
Pregled Fourierovih transformacija

Vremenski kontinuirana Fourierova transformacija (CTFT)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$E = \int_{-\infty}^{+\infty} x(t)x^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)X^*(j\omega)\,d\omega$$

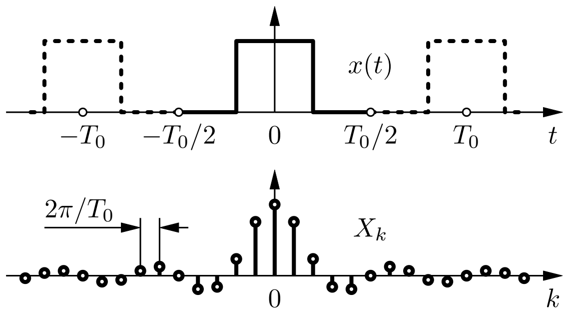


Vremenski kontinuiran Fourierov red (CTFS)

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{j\omega_0 kt}$$

$$P = \frac{1}{T_0}\int_{T_0} x(t)x^*(t)\,dt = \sum_{k=-\infty}^{+\infty} X_k X_k^*$$

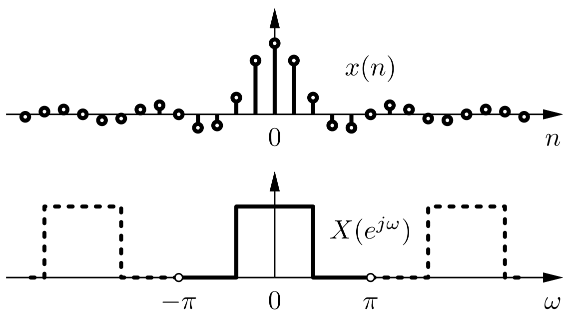


Vremenski diskretna Fourierova transformacija (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$E = \sum_{n=-\infty}^{+\infty} x(n)x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})X^*(e^{j\omega})\,d\omega$$

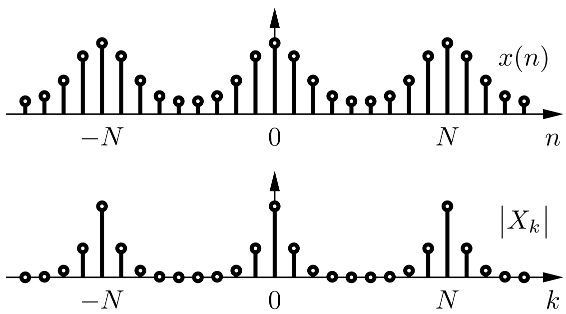


Vremenski diskretan Fourierov red (DTFS)

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-2\pi jkn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{2\pi jkn/N}$$

$$P = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x^*(n) = \sum_{k=0}^{N-1} X_k X_k^*$$

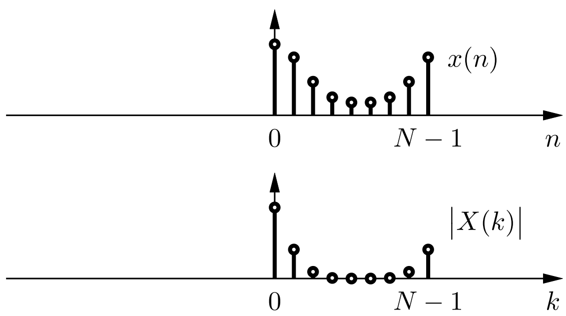


Diskretna Fourierova transformacija (DFT)

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \le n \le N-1$$

$$\sum_{n=0}^{N-1} x_1(n)x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k)$$



Određivanja početnih uvjeta

Za sustav opisan diferencijalnom jednadžbom

$$a_0 y^{(N)}(t) + a_1 y^{(N-1)}(t) + \cdots + a_{N-1} y^{(1)}(t) + a_N y(t) = b_0 u^{(N)}(t) + b_1 u^{(N-1)}(t) + \cdots + b_{N-1} u^{(1)}(t) + b_N u(t)$$

uz $a_0 \neq 0$ potrebno je odrediti početne uvjete $y(0^+)$, $y'(0^+)$, $y''(0^+)$, …, $y^{(N-1)}(0^+)$ u 0^+ iz onih u 0^-.

Ako je pobuda glatka funkcija onda su početni uvjeti u 0^+ i 0^- jednaki.

Ako pobuda glatka svugdje osim u nuli gdje ima konačan skok tada rješavamo sustav jednadžbi:

$$\begin{aligned}
a_0 \Delta y &= b_0 \Delta u \\
a_0 \Delta y^{(1)} + a_1 \Delta y &= b_0 \Delta u^{(1)} + b_1 \Delta u \\
a_0 \Delta y^{(2)} + a_1 \Delta y^{(1)} + a_2 \Delta y &= b_0 \Delta u^{(2)} + b_1 \Delta u^{(1)} + b_2 \Delta u \\
\\
a_0 \Delta y^{(N-1)} + \cdots + a_{N-2} \Delta y^{(1)} + a_{N-1} \Delta y &= b_0 \Delta u^{(N-1)} + \cdots + b_{N-2} \Delta u^{(1)} + b_{N-1} \Delta u
\end{aligned}$$

Ako pobuda dodatno uz konačan skok u nuli sadrži i Diracovu distribuciju $k\delta(t)$, gdje je k realna konstanta, tada rješavamo sustav jednadžbi:

$$\begin{aligned}
a_0 \Delta y + a_1 k \frac{b_0}{a_0} &= b_0 \Delta u + k b_1 \\
a_0 \Delta y^{(1)} + a_1 \Delta y + a_2 k \frac{b_0}{a_0} &= b_0 \Delta u^{(1)} + b_1 \Delta u + k b_2 \\
a_0 \Delta y^{(2)} + a_1 \Delta y^{(1)} + a_2 \Delta y + a_3 k \frac{b_0}{a_0} &= b_0 \Delta u^{(2)} + b_1 \Delta u^{(1)} + b_2 \Delta u + k b_3 \\
\\
a_0 \Delta y^{(N-1)} + \cdots + a_{N-2} \Delta y^{(1)} + a_{N-1} \Delta y + a_N k \frac{b_0}{a_0} &= b_0 \Delta u^{(N-1)} + \cdots + b_{N-2} \Delta u^{(1)} + b_{N-1} \Delta u + k b_N
\end{aligned}$$

Pri tome je $\Delta y^{(i)} = y^{(i)}(0^+) - y^{(i)}(0^-)$ i $\Delta u^{(i)} = u^{(i)}(0^+) - u^{(i)}(0^-)$.