

# TACTICAL DECOMPOSITIONS OF DESIGNS OVER FINITE FIELDS

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ABSTRACT. An automorphism group of an incidence structure  $\mathcal{I}$  induces a tactical decomposition on  $\mathcal{I}$ . It is well known that tactical decompositions of  $t$ -designs satisfy certain necessary conditions which can be expressed as equations in terms of the coefficients of tactical decomposition matrices. In this article we present results obtained for tactical decompositions of  $q$ -analogs of  $t$ -designs, more precisely, of  $2$ - $(v, k, \lambda_2; q)$  designs. We show that coefficients of tactical decomposition matrices of a design over finite field satisfy an equation system analog to the one known for block designs. Furthermore, taking into consideration specific properties of designs over the binary field  $\mathbb{F}_2$ , we obtain an additional system of inequations for these coefficients in that case.

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## 1. INTRODUCTION AND PRELIMINARY RESULTS

Let  $\mathbb{F}_q$  be the finite field of order  $q$  and  $\mathbb{F}_q^v$  the vector space of dimension  $v$  over the finite field  $\mathbb{F}_q$ . An  $r$ -space is a subspace of  $\mathbb{F}_q^v$  of dimension  $r$ . The number of  $r$ -spaces of  $\mathbb{F}_q^v$  is

$$\begin{bmatrix} v \\ r \end{bmatrix}_q = \frac{(q^v - 1) \cdots (q^{v-r+1} - 1)}{(q^r - 1) \cdots (q - 1)}.$$

The number of  $r$ -spaces containing a fixed  $s$ -space,  $s \leq r$ , is

$$\begin{bmatrix} v - s \\ r - s \end{bmatrix}_q.$$

For every two subspaces  $U$  and  $V$ , the *dimension formula* holds:

$$\dim \langle U, V \rangle = \dim U + \dim V - \dim (U \cap V).$$

Designs over finite fields were first introduced in the 1970's, see [4],[5],[6]. First nontrivial designs over finite fields which are not spreads were constructed in [14].

**Definition 1.1.** A finite set  $\mathcal{B}$  is called *design over finite field with parameters  $t$ - $(v, k, \lambda_t)$*  if the following properties hold:

- (1) elements of  $\mathcal{B}$  are  $k$ -spaces of the vector space  $\mathbb{F}_q^v$  called blocks,
- (2) every  $t$ -space of  $\mathbb{F}_q^v$  is contained in  $\lambda_t$  blocks.

Designs over finite fields are also often called  $q$ -analogs of  $t$ -designs, or shorter  $q$ -designs. A  $t$ - $(v, k, \lambda_t)$  design is a finite incidence structure  $(\mathcal{P}, \mathcal{B})$ , where  $\mathcal{P}$  is a set of  $v$  elements called *points*, and  $\mathcal{B}$  is a multiset of nonempty  $k$ -subsets of  $\mathcal{P}$  called *blocks* such that every set of  $t$  distinct points is contained in exactly  $\lambda_t$  blocks. When parameters are not important,  $t$ - $(v, k, \lambda_t)$  designs are shorter called  $t$ -designs. When  $t = 2$ , designs are called *block designs*. Designs over finite fields are closely related to  $t$ -designs. Every design  $\mathcal{B}$  with parameters  $2$ - $(v, k, \lambda_2; q)$  gives a block design with parameters  $2$ - $(\begin{smallmatrix} v \\ 1 \end{smallmatrix}_q, \begin{smallmatrix} k \\ 1 \end{smallmatrix}_q, \lambda_2)$ , where points are identified with 1-spaces of  $\mathbb{F}_q^v$  and each block is identified with the set of 1-spaces it contains. The inverse statement is not valid. For example, there are block designs with parameters  $2$ - $(15, 7, 3)$  which cannot be constructed from the associated  $2$ - $(4, 3, 3; 2)$  design.

If  $\mathcal{B}$  is a design with parameters  $t$ - $(v, k, \lambda_t; q)$ , then  $\mathcal{B}$  is a design with parameters  $s$ - $(v, k, \lambda_s; q)$ ,  $0 \leq s \leq t$ , where

$$\lambda_s = \lambda_t \frac{\begin{smallmatrix} v-s \\ t-s \end{smallmatrix}_q}{\begin{smallmatrix} k-s \\ t-s \end{smallmatrix}_q}.$$

The number of blocks in  $\mathcal{B}$  equals

$$|\mathcal{B}| = \lambda_t \frac{\begin{smallmatrix} v \\ t \end{smallmatrix}_q}{\begin{smallmatrix} k \\ t \end{smallmatrix}_q}.$$

*Automorphism* of  $\mathcal{B}$  is a linear operator  $\Phi \in GL_v(q)$  such that  $\Phi\mathcal{B} = \mathcal{B}$ . The set  $Aut \mathcal{B}$  of all automorphisms of  $\mathcal{B}$  is a subgroup of the general linear group  $GL_v(q)$ , called *full automorphism group* of  $\mathcal{B}$ . Any subgroup of  $Aut \mathcal{B}$  is an automorphism group of  $\mathcal{B}$ .

## 2. TACTICAL DECOMPOSITIONS OF DESIGNS OVER FINITE FIELDS

The idea of considering tactical decompositions of block designs was first introduced by Dembowski [7]. Equations for coefficients of tactical decomposition matrices for block designs are well known [9] and they were used for constructions of many examples of block designs (listed in [13]). These equations were generalized for any  $t \geq 1$  in [12]. In this article we introduce tactical decompositions of designs over finite fields for  $t = 2$ . We show that coefficients of tactical decomposition matrices satisfy an equation system analog to the one known for block designs. Furthermore, taking into consideration specific properties of designs over the binary field  $\mathbb{F}_2$ , we obtain an additional system of inequations

for coefficients of tactical decomposition matrices of a  $2-(v, k, \lambda_2; 2)$  design. The system of equations and inequations for coefficients of tactical decomposition matrices represents necessary conditions for the existence of designs over finite fields with an assumed automorphism group.

The Kramer-Mesner method [10] has been adopted and used for construction of designs over finite fields, see [1], [2], [3]. In [11] it was introduced how a tactical decomposition of a  $t$ -design induced by an action of a proposed automorphism group can be used for the enhancement of the Kramer-Mesner method. The necessary conditions on the existence of designs over finite fields with an assumed automorphism group introduced in this article can be implemented in the Kramer-Mesner method for construction of designs over finite fields, in a manner analog to [11].

**Definition 2.1.** *Let  $\Psi$  be the set of all 1-spaces of a finite vector space  $V$  over a finite field  $\mathbb{F}$ . Elements of  $\Psi$  shall be called points. A decomposition of a design  $\mathcal{B}$  over a finite field  $\mathbb{F}$  is a partition of the set of points  $\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m$  and the set  $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ . We say that a decomposition is tactical if there exist nonnegative integers  $\rho_{ij}, \kappa_{ij}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ , such that*

- (1) every point in  $\Psi_i$  is contained in  $\rho_{ij}$  blocks in  $\mathcal{B}_j$ ,
- (2) each block in  $\mathcal{B}_j$  contains  $\kappa_{ij}$  points in  $\Psi_i$ .

Matrices  $[\rho_{ij}]$  and  $[\kappa_{ij}]$  are called tactical decomposition matrices.

There are two trivial examples of tactical decomposition of a design. The first example is obtained by putting  $n = m = 1$ , and the second by partitioning sets  $\Psi$  and  $\mathcal{B}$  into 1-element subsets. A nontrivial tactical decomposition can be obtained by an action of an automorphism group  $G \leq \text{Aut}(\mathcal{B})$  on a design.

**Theorem 2.2.** *Let  $G$  be an automorphism group of a design over finite field  $\mathcal{B}$ . Then the orbits of the set of points  $\Psi$  and the orbits of  $\mathcal{B}$  form a tactical decomposition.*

Let  $\mathcal{B}$  be a design with parameters  $2-(v, k, \lambda_2; q)$ . Let

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n,$$

be a tactical decomposition of  $\mathcal{B}$ . For  $P \in \Psi$  we denote by  $\mathcal{I}_P = \{B \in \mathcal{B} \mid P \leq B\}$  the set of all blocks containing  $P$ . Obviously,  $|\mathcal{I}_P| = \lambda_1$  and

$$\begin{aligned} \rho_{ij} &= |\mathcal{I}_P \cap \mathcal{B}_j|, \quad P \in \Psi_i, \\ \kappa_{ij} &= \left| \begin{bmatrix} B \\ 1 \end{bmatrix} \cap \Psi_i \right|, \quad B \in \mathcal{B}_j, \end{aligned}$$

where  $\left[ \begin{smallmatrix} B \\ 1 \end{smallmatrix} \right]$  is the set of all 1-spaces of  $B$ . Coefficients  $\rho_{ij}$  and  $\kappa_{ij}$  are not dependant on the choice of  $P \in \Psi_i$  and of  $B \in \mathcal{B}_j$  if and only if the decomposition is tactical. It is easy to show that

$$(1) \quad \begin{aligned} \sum_{i=1}^m \kappa_{ij} &= \left[ \begin{smallmatrix} k \\ 1 \end{smallmatrix} \right]_q, \\ \sum_{j=1}^n \rho_{ij} &= \lambda_1. \end{aligned}$$

Double-counting of the set  $\{(P, B) \in \Psi_i \times \mathcal{B}_j : P \leq B\}$  yields

$$(2) \quad |\Psi_i| \cdot \rho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij}.$$

Now, fix a point  $P \in \Psi_l$ . Double-counting of the set

$$\{(Q, B) \in \Psi_r \times \mathcal{B} : P, Q \leq B\}$$

yields

$$(3) \quad \sum_{j=1}^n \rho_{lj} \kappa_{rj} = \sum_{Q \in \Psi_r} |\mathcal{I}_P \cap \mathcal{I}_Q|.$$

It is easy to compute the right-hand side of the previous expression. Obviously,  $\mathcal{I}_P \cap \mathcal{I}_Q = \{B \in \mathcal{B} : \langle P, Q \rangle \leq B\}$  and so

$$|\mathcal{I}_P \cap \mathcal{I}_Q| = \begin{cases} \lambda_1, & P = Q, \\ \lambda_2, & P \neq Q. \end{cases}$$

Thus, we have obtained a system of equations for the coefficients of tactical decomposition matrices.

**Theorem 2.3.** *Assume  $\mathcal{B}$  is a  $2$ -( $v, k, \lambda_2; q$ ) design with a tactical decomposition*

$$\Psi = \Psi_1 \sqcup \dots \sqcup \Psi_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_n.$$

*Let  $[\rho_{ij}]$  and  $[\kappa_{ij}]$  be the associated tactical decomposition matrices. Then*

$$(4) \quad \sum_{j=1}^n \rho_{lj} \kappa_{rj} = \begin{cases} \lambda_2 \cdot |\Psi_r|, & l \neq r, \\ \lambda_1 + \lambda_2 \cdot (|\Psi_r| - 1), & l = r. \end{cases}$$

### 3. IMPROVEMENTS FOR THE BINARY FIELD

Assume now that  $\mathcal{B}$  is a design over the binary field  $\mathbb{F}_2$  with parameters  $2$ -( $v, k, \lambda_2; 2$ ) and with an automorphism group  $G \leq GL_v(2)$ . Then the orbits of  $\Psi$  and the orbits of  $\mathcal{B}$  form a tactical decomposition

$$\Psi = \Psi_1 \sqcup \dots \sqcup \Psi_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_n.$$

Fix a point  $P \in \Psi_l$ . Double-counting of the set

$$\{(R, S, B) \in \Psi_r \times \Psi_s \times \mathcal{B} : P, R, S \leq B\}$$

yields

$$(5) \quad \sum_{j=1}^m \rho_{lj} \kappa_{rj} \kappa_{sj} = \sum_{R \in \Psi_r} \sum_{S \in \Psi_s} |\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S|.$$

Let  $R \in \Psi_r, S \in \Psi_s$ . Obviously

$$\mathcal{I}_P \cap \mathcal{I}_R \cap \mathcal{I}_S = \{B \in \mathcal{B} : \langle P, R, S \rangle \leq B\} =: \mathcal{I}_{PRS}.$$

By the dimension formula,  $1 \leq \dim \langle P, R, S \rangle \leq 3$ . For  $R \in \Psi_r$ , let  $\Psi_s^i(R)$  be the set of all  $Q \in \Psi_s$  such that  $\dim \langle P, R, Q \rangle = i$ ,  $i = 1, 2, 3$ ,

$$\Psi_s^i(R) := \{Q \in \Psi_s : \dim \langle P, R, Q \rangle = i\}.$$

Sets  $\Psi_s^i(R)$  are pairwise disjoint and form a partition of  $\Psi_s$ ,

$$\Psi_s = \Psi_s^1(R) \sqcup \Psi_s^2(R) \sqcup \Psi_s^3(R).$$

Let

$$\phi_{rs}^i = \sum_{R \in \Psi_r} \sum_{S \in \Psi_s^i(R)} |\mathcal{I}_{PRS}|, \quad i = 1, 2, 3.$$

Then

$$(6) \quad \sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \phi_{rs}^1 + \phi_{rs}^2 + \phi_{rs}^3.$$

In the continuation, we compute  $\phi_{rs}^1, \phi_{rs}^2$ , and obtain an upper bound for  $\phi_{rs}^3$ . It is easy to see that

$$|\mathcal{I}_{PRS}| = \begin{cases} \lambda_1, & S \in \Psi_s^1(R), \\ \lambda_2, & S \in \Psi_s^2(R). \end{cases}$$

For  $S \in \Psi_s^3(R)$  we can obtain only an upper bound,

$$(7) \quad |\mathcal{I}_{PRS}| \leq \min\{\lambda_2, \begin{bmatrix} v-3 \\ k-3 \end{bmatrix}_q\} =: \varphi.$$

Consequently, we can obtain only an upper bound for the right-hand side of (6).

We denote the 3 points of a 2-space  $\langle P, R \rangle$  of  $\mathbb{F}_2^v$  by  $P, R$  and  $P + R$ . Let  $\mathcal{M}_{rs}(P) \subseteq \Psi_r$  be the set of all points  $R \in \Psi_r, P \neq R$ , such that  $P + R \in \Psi_s$ ,

$$\mathcal{M}_{rs}(P) := \{R \in \Psi_r \setminus \{P\} : P + R \in \Psi_s\}.$$

Tactical decomposition of  $\mathcal{B}$  is group-induced. Hence, the cardinality of  $\mathcal{M}_{rs}(P)$  is not dependant of the choice of  $P \in \Psi_l$ , i.e.  $|\mathcal{M}_{rs}(P)| = |\mathcal{M}_{rs}(P')|, \forall P' \in \Psi_l$ . We shall write  $\sigma_{lrs} := |\mathcal{M}_{rs}(P)|$ . The cardinality

of  $\Psi_s^i(R)$ ,  $i = 1, 2, 3$ , varies depending on whether  $R = P$ ,  $R \in \mathcal{M}_{rs}(P)$  or otherwise.

**Lemma 3.1.** *Assume  $\mathcal{B}$  is a  $2$ - $(v, k, \lambda_2; 2)$  design with an automorphism group  $G$  and a  $G$ -induced tactical decomposition*

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Let  $P \in \Psi_l$  and  $R \in \Psi_r$ . Then

$$|\Psi_s^1(R)| = \begin{cases} 1, & l = r = s \text{ i } R = P, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore:

(1) For  $l \neq r \neq s \neq l$  holds

$$|\Psi_s^2(R)| = \begin{cases} 1, & R \in \mathcal{M}_{rs}(P), \\ 0, & \text{otherwise,} \end{cases}$$

$$|\Psi_s^3(R)| = \begin{cases} |\Psi_s| - 1, & R \in \mathcal{M}_{rs}(P), \\ |\Psi_s|, & \text{otherwise.} \end{cases}$$

(2) For  $l = r = s$  holds

$$|\Psi_s^2(R)| = \begin{cases} 3, & R \in \mathcal{M}_{rs}(P), \\ |\Psi_s| - 1, & R = P, \\ 2, & \text{otherwise,} \end{cases}$$

$$|\Psi_s^3(R)| = \begin{cases} |\Psi_s| - 3, & R \in \mathcal{M}_{rs}(P), \\ 0, & R = P, \\ |\Psi_s| - 2, & \text{otherwise.} \end{cases}$$

(3) For  $l = r \neq s$  holds

$$|\Psi_s^2(R)| = \begin{cases} 1, & R \in \mathcal{M}_{rs}(P), \\ |\Psi_s|, & R = P, \\ 0, & \text{otherwise,} \end{cases}$$

$$|\Psi_s^3(R)| = \begin{cases} |\Psi_s| - 1, & R \in \mathcal{M}_{rs}(P), \\ 0, & R = P, \\ |\Psi_s|, & \text{otherwise.} \end{cases}$$

(4) For  $l \neq r = s$  holds

$$|\Psi_s^2(R)| = \begin{cases} 2, & R \in \mathcal{M}_{rs}(P), \\ 1, & \text{otherwise,} \end{cases}$$

$$|\Psi_s^3(R)| = \begin{cases} |\Psi_s| - 2, & R \in \mathcal{M}_{rs}(P), \\ |\Psi_s| - 1, & \text{otherwise.} \end{cases}$$

□

*Proof.* It is easy to see that

$$\Psi_s^1(R) = \begin{cases} \{P\}, & l = r = s \text{ and } R = P, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We shall now determine  $\Psi_s^2(R)$  in each of the four cases. Then,

$$\Psi_s^3(R) = \Psi_s \setminus (\Psi_s^1(R) \sqcup \Psi_s^2(R)).$$

Note that for  $R \in \Psi_r$  holds

$$\Psi_s^2(R) \subseteq \{P, R, P + R\}.$$

Let  $l \neq r \neq s \neq l$ . Then  $P, R \notin \Psi_s$ , hence

$$\Psi_s^2(R) = \begin{cases} \{P + R\}, & R \in \mathcal{M}_{rs}(P), \\ \emptyset, & R \notin \mathcal{M}_{rs}(P). \end{cases}$$

Let  $l = r = s$ . Then  $P, R \in \Psi_l$  and

$$\Psi_s^2(R) = \begin{cases} \{P, R, P + R\}, & R \in \mathcal{M}_{rs}(P), \\ \Psi_s \setminus \{P\}, & R = P, \\ \{P, R\}, & \text{otherwise.} \end{cases}$$

Let  $l = r \neq s$ . Then

$$\Psi_s^2(R) = \begin{cases} \{P + R\}, & R \in \mathcal{M}_{rs}(P), \\ \Psi_s, & R = P, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let  $l \neq r = s$ . Then

$$\Psi_s^2(R) = \begin{cases} \{R, P + R\}, & R \in \mathcal{M}_{rs}(P), \\ \{R\}, & \text{otherwise.} \end{cases}$$

□

The following theorem gives additional necessary conditions on the existence of a  $2-(v, k, \lambda_2; 2)$  design with an assumed automorphism group.

**Theorem 3.2.** *Assume  $\mathcal{B}$  is a  $2$ - $(v, k, \lambda_2; 2)$  design with an automorphism group  $G$  and a  $G$ -induced tactical decomposition*

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Let  $[\rho_{ij}]$  and  $[\kappa_{ij}]$  be the associated tactical decomposition matrices. Then

$$\begin{aligned} & \sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \\ & = \begin{cases} \sigma_{lrs} \cdot \lambda_2 + \phi_{rs}^3, & l \neq r \neq s \neq l, \\ \lambda_1 + (3|\Psi_l| + \sigma_{lrs} - 3)\lambda_2 + \phi_{rs}^3, & l = r = s, \\ (|\Psi_s| + \sigma_{lrs})\lambda_2 + \phi_{rs}^3, & l = r \neq s \text{ or } l \neq r = s, \end{cases} \end{aligned}$$

and

$$\phi_{rs}^3 \leq \begin{cases} (|\Psi_r| \cdot |\Psi_s| - \sigma_{lrs})\varphi, & l \neq r \neq s \neq l, \\ (|\Psi_l|^2 - 3|\Psi_l| - \sigma_{lrs} + 2)\varphi, & l = r = s, \\ (|\Psi_r| \cdot |\Psi_s| - |\Psi_s| - \sigma_{lrs})\varphi, & l = r \neq s \text{ or } l \neq r = s, \end{cases}$$

where  $\varphi := \min\{\lambda_2, \lfloor \frac{v-3}{k-3} \rfloor\}$ .

*Proof.* Fix a point  $P \in \Psi_l$ . Then it holds

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \phi_{rs}^1 + \phi_{rs}^2 + \phi_{rs}^3.$$

Applying Lemma 3.1, it is easy to compute  $\phi_{rs}^1$  and  $\phi_{rs}^2$ , and obtain an upper bound for  $\phi_{rs}^3$ .

Let  $l \neq r \neq s \neq l$ . Then  $\Psi_r = \mathcal{M}_{rs}(P) \sqcup \overline{\mathcal{M}}_{rs}(P)$ ,

$$\phi_{rs}^2 = \sum_{R \in \Psi_r} \sum_{S \in \Psi_s^2(R)} \lambda_2 = \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 = \sigma_{lrs} \cdot \lambda_2,$$

while

$$\begin{aligned} \phi_{rs}^3 &= \sum_{R \in \Psi_r} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| \\ &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| \\ &\leq \sigma_{lrs}(|\Psi_s| - 1)\varphi + (|\Psi_r| - \sigma_{lrs})|\Psi_s|\varphi \\ &= (|\Psi_r| \cdot |\Psi_s| - \sigma_{lrs})\varphi. \end{aligned}$$

Let  $l = r = s$ . Then  $\Psi_r = \mathcal{M}_{rs}(P) \sqcup \{P\} \sqcup \overline{\mathcal{M}}_{rs}(P)$ ,

$$\begin{aligned}\phi_{rs}^2 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 + \sum_{S \in \Psi_s^2(P)} \lambda_2 \\ &= 3\sigma_{lrs}\lambda_2 + 2(|\Psi_l| - \sigma_{lrs} - 1)\lambda_2 + (|\Psi_l| - 1)\lambda_2 \\ &= (3|\Psi_l| + \sigma_{lrs} - 3)\lambda_2,\end{aligned}$$

while

$$\begin{aligned}\phi_{rs}^3 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| \\ &\leq \sigma_{lrs}(|\Psi_s| - 3)\varphi + (|\Psi_r| - \sigma_{lrs} - 1)(|\Psi_s| - 2)\varphi \\ &= (|\Psi_l|^2 - 3|\Psi_l| - \sigma_{lrs} + 2)\varphi.\end{aligned}$$

Let  $l = r \neq s$ . Then  $\Psi_l = \mathcal{M}_{rs}(P) \sqcup \{P\} \sqcup \overline{\mathcal{M}}_{rs}(P)$ ,

$$\begin{aligned}\phi_{rs}^2 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 + \sum_{S \in \Psi_s^2(P)} \lambda_2 \\ &= (\sigma_{lrs} + |\Psi_s|)\lambda_2,\end{aligned}$$

while

$$\begin{aligned}\phi_{rs}^3 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| \\ &\leq (|\Psi_s| - 1)\sigma_{lrs}\varphi + (|\Psi_r| - \sigma_{lrs} - 1)|\Psi_s|\varphi \\ &= (|\Psi_r| \cdot |\Psi_s| - |\Psi_s| - \sigma_{lrs})\varphi.\end{aligned}$$

Let  $l \neq r = s$ . Then  $\Psi_r = \mathcal{M}_{rs}(P) \sqcup \overline{\mathcal{M}}_{rs}(P)$ ,

$$\begin{aligned}\phi_{rs}^2 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^2(R)} \lambda_2 \\ &= 2\sigma_{lrs}\lambda_2 + (|\Psi_s| - \sigma_{lrs})\lambda_2 \\ &= (|\Psi_s| + \sigma_{lrs})\lambda_2,\end{aligned}$$

while

$$\begin{aligned}\phi_{rs}^3 &= \sum_{R \in \mathcal{M}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| + \sum_{R \in \overline{\mathcal{M}}_{rs}(P)} \sum_{S \in \Psi_s^3(R)} |\mathcal{I}_{PRS}| \\ &\leq \sigma_{lrs}(|\Psi_s| - 2)\varphi + (|\Psi_r| - \sigma_{lrs})(|\Psi_s| - 1)\varphi \\ &= (|\Psi_r| \cdot |\Psi_s| - |\Psi_r| - \sigma_{lrs})\varphi.\end{aligned}$$

□

□

Note that the application of the equality

$$\kappa_{ij} = \frac{|\Psi_i|}{|\mathcal{B}_j|} \rho_{ij}$$

on the left-hand side of the expression (6) eliminates the coefficients  $\kappa_{ij}$  from (6) and yields a system of inequations for the coefficients  $\rho_{ij}$

$$\sum_{j=1}^n \rho_{lj} \kappa_{rj} \kappa_{sj} = \sum_{j=1}^n \frac{|\Psi_r| \cdot |\Psi_s|}{|\mathcal{B}_j|^2} \rho_{lj} \rho_{rj} \rho_{sj}.$$

An analog relation is valid for the coefficients  $\kappa_{ij}$  as well.

#### 4. EXAMPLES FOR SOME CYCLIC GROUPS

We shall now illustrate our results on the example of a design  $\mathcal{B}$  with parameters 2-(4, 3, 3; 2). The obtained design is not new, however this example contours well the scope of our results. Let  $G = \langle \Phi \rangle \leq GL_4(2)$ ,

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Group  $G$  is the cyclic group of order 3. Assume  $G$  is an automorphism group of  $\mathcal{B}$ . Then

$$\Psi = \Psi_1 \sqcup \dots \sqcup \Psi_5,$$

with respective orbit representatives  $\langle [1, 0, 0, 0] \rangle$ ,  $\langle [1, 0, 1, 0] \rangle$ ,  $\langle [1, 0, 1, 1] \rangle$ ,  $\langle [1, 1, 0, 0] \rangle$ ,  $\langle [0, 1, 0, 0] \rangle$ . All orbits are of length 3. The orbits of  $\mathcal{B}$  are currently unknown to us, but it is obvious that these orbits are of length 3. In addition, the orbits of  $\Psi$  and the orbits of  $\mathcal{B}$  form a tactical decomposition

$$\Psi = \Psi_1 \sqcup \dots \sqcup \Psi_5, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_5.$$

Coefficients of a corresponding tactical decomposition matrix  $[\rho_{ij}]$  must satisfy the equations (1) and (4),

$$\sum_{j=1}^5 \rho_{ij} = 7, \quad i = 1, \dots, 5,$$

$$\sum_{i=1}^5 \rho_{ij} = 7, \quad j = 1, \dots, 5,$$

$$\sum_{j=1}^5 \rho_{ij}^2 = 13, \quad i = 1, \dots, 5,$$

$$\sum_{j=1}^5 \rho_{rj} \rho_{sj} = 9, \quad r \neq s.$$

There are precisely two matrices, up to a rearrangement of rows and columns, with coefficients that satisfy the above mentioned equations:

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 \end{bmatrix}.$$

Note that the coefficients of the tactical decomposition matrices of a block design with corresponding parameters  $2-(15, 7, 3)$  also necessarily satisfy this system of equations.

Furthermore, by the Theorem 3.2, the coefficients  $\rho_{ij}$  satisfy an additional system of inequations. First we determine the values  $\sigma_{lrs}$ ,

$$\sigma_{lrs} = \begin{cases} 1, & l \neq r \neq s \neq l, \\ 2, & l = r = s, \\ 0, & l = r \neq s \text{ or } l \neq r = s. \end{cases}$$

Coefficients  $\rho_{ij}$  necessarily satisfy these inequations:

$$(8) \quad 31 \leq \sum_{j=1}^5 \rho_{lj}^3 \leq 31, \quad l = 1, \dots, 5,$$

$$(9) \quad 3 \leq \sum_{j=1}^5 \rho_{lj} \rho_{rj} \rho_{sj} \leq 11, \quad l \neq r \neq s \neq l,$$

$$(10) \quad 9 \leq \sum_{j=1}^5 \rho_{lj}^2 \rho_{rj} \leq 15, \quad l \neq r.$$

Only one of the two constructed matrices satisfies these additional constraints. For the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 \end{bmatrix},$$

it holds that

$$\sum_{j=1}^5 \rho_{2j}^3 = 25,$$

a contradiction with inequation (8). Hence, the associated tactical decomposition matrix  $[\rho_{ij}]$  of a design with parameters  $2-(4, 3, 3; 2)$  and automorphism group  $G$ , equals to the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix},$$

up to a rearrangement of rows and columns. There is a unique  $2-(4, 3, 3; 2)$  design. It can be obtained by taking all the hyperplanes of the projective space  $\text{PG}(3, 2)$ . In general, the coefficients of the tactical decomposition matrices of a block design with the corresponding parameters do not necessarily satisfy the system of inequations from Theorem 3.2. Namely, there are block designs with parameters  $2-(15, 7, 3)$  for each of the two constructed tactical decomposition matrices. For the computation of the matrices we used the program `orbmat5qd` made by V. Krčadinac [11], application GAP [8] and our own programs.

Hereafter, we give another example. Consider now a design  $\mathcal{B}$  with parameters  $2-(6, 3, 6; 2)$ . Let  $\Phi \in GL_6(2)$ ,

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then  $G = \langle \Phi \rangle \leq GL_6(2)$  is the cyclic group of order 31. Assume that  $G$  is an automorphism group of  $\mathcal{B}$ . Then

$$\Psi = \Psi_1 \sqcup \Psi_2 \sqcup \Psi_3,$$

with respective orbit representatives  $\langle [1, 1, 1, 1, 0, 1] \rangle$ ,  $\langle [1, 0, 0, 0, 0, 0] \rangle$ ,  $\langle [1, 0, 0, 0, 0, 1] \rangle$ . Moreover,  $|\Psi_1| = 1$ ,  $|\Psi_2| = |\Psi_3| = 31$ . Furthermore, all orbits of  $\mathcal{B}$ , currently unknown to us, are necessarily of length 31, and the orbits of  $\Psi$  and the orbits of  $\mathcal{B}$  form a tactical decomposition

$$\Psi = \Psi_1 \sqcup \Psi_2 \sqcup \Psi_3, \quad \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_{18}.$$

In addition,

$$\sigma_{1rs} = \begin{cases} 31, & 1 \neq r \neq s \neq 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_{2rs} = \begin{cases} 30, & r \neq s, r, s = 2, 3, \\ 1, & 2 \neq r \neq s \neq 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_{3rs} = \begin{cases} 30, & r = s, r, s = 2, 3, \\ 1, & 3 \neq r \neq s \neq 3, \\ 0, & \text{otherwise.} \end{cases}$$

We constructed 65 matrices satisfying the equations (1) and (4) for coefficients  $\rho_{ij}$  of tactical decomposition matrices. Out of these 65 matrices, 3 do not satisfy the system of inequations from Theorem 3.2:

$$\begin{bmatrix} 31 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 6 & 0 & 6 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 31 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 6 & 0 & 5 & 5 & 5 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 31 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 \\ 0 & 6 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 \end{bmatrix}.$$

Namely, the coefficients of these 3 matrices do not satisfy the inequality

$$186 \leq \sum_{j=1}^{18} \rho_{1j} \rho_{2j} \rho_{3j} \leq 1116.$$

For each of the remaining 62 matrices we attempted to construct a design over finite field with parameters  $2-(6, 3, 6; 2)$ , automorphism group  $G$  and associated tactical decomposition matrix  $M$ . For the construction we used a method analog to the one described in [11]. We conclude that such design exists only when  $M$  is

$$\begin{bmatrix} 31 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 3 & 3 & 7 & 7 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

Finally, we could construct 330 designs with given parameters, admitting the action of the cyclic group  $G$  of order 31.

In [1] examples of  $2-(6, 3, 6; 2)$  were constructed. The author used a different automorphism group  $G = \langle \sigma^7 \rangle$ , where  $\sigma$  is the Singer cycle,

hence,  $G$  is the cyclic group of order 9. The number of constructed designs is not reported.

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