Non-existence of a simple 3-$(16, 7, 5)$ design with an automorphism of order 3

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Abstract

Using recent results concerning tactical decompositions of t-designs with $t > 2$, we make a step forward on the long-standing question about the existence of a simple 3-$(16, 7, 5)$ design; if such a design exists, then its full automorphism group has order a power of 2, possibly equal to 1.

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1. Introduction

The smallest v for which the existence of a 3-design of order v is undecided is 16; indeed a 3-$(16, 7, 5)$ design is still unknown [10]. Thus to solve this intriguing existence problem has turned out to be a challenge.

So far, the published results on this problem bring negative answers if some additional properties on the automorphism group of the desired design are assumed. In this article we also answer in the negative if we wanted to prescribe an automorphism of order three.

In [3] Z. Eslami showed that a simple 3-$(16, 7, 5)$ design with an automorphism of prime order $p \geq 5$ does not exist. This result was obtained by determining, up to isomorphism, all 2-$(15, 6, 5)$ designs possessing an automorphism of prime order $p \geq 5$ and then showing that none of these 1454 designs can be the derived design of a 3-$(16, 7, 5)$ design.

At this moment, to classify all 2-$(15, 6, 5)$ designs with an automorphism of order 3 seems to be unfeasible. Thus, for extending Eslami result to $p \geq 3$ we had to follow a new strategy. Indeed our proof is based on *tactical*

decompositions [2]. They have been crucial for the construction of many 2-designs [5, 8], but we are not aware of existence (or non-existence) results about t-designs with $t > 2$ obtained via them. The present article allowed the author to show the effectiveness of the equations for coefficients of tactical decomposition matrices obtained in [8, 9]. Indeed they have been the key tool for the main result. Note that we are now able to state the following theorem.

Theorem 1.1. *If a simple* 3-$(16, 7, 5)$ *design exists, then the order of its full automorphism group is a power of* 2.

We point out that our technique might also be used for proving that a putative $3-(16, 7, 5)$ design $\mathcal{D}$ is necessarily rigid. For this, it would be enough to show that the system of equations arising from the tactical decomposition associated with an automorphism of $\mathcal{D}$ of order 2 leads to an absurd. On the other hand we expect that the computations are extremely demanding in view of the larger sizes of the corresponding tactical decomposition matrix $\mathcal{K}$. We also point out that similar arguments could be applied to get informations on the full automorphism group of other t-designs with $t > 2$. The most natural thing would be to consider a 3-$(17, 7, 7)$ design whose existence is also in doubt (see Remark 4.45 in [10]). Here the reason for which we also expect too demanding computations is that the number of blocks, that is 136, is rather larger than the number of blocks of a 3-$(16, 7, 5)$ design.

2. Preliminary results

Let t, v, k, λ_t be positive integers with $v > k \geq t$. A t-(v, k, λ_t) *design* is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v elements called *points*, and $\mathcal{B}$ is a multiset of k-subsets of $\mathcal{P}$ called *blocks* such that every set of t distinct points is contained in precisely λ_t blocks. A design is said to be *simple* if there are no repeated blocks. One says that a point $P \in \mathcal{P}$ is *incident* with a block $B \in \mathcal{B}$ if $P \in B$. The set of all blocks of $\mathcal{D}$ containing a given set $\mathcal{S}$ of points will be denoted by $\mathcal{I}_{\mathcal{S}}$. If $\mathcal{S} = \{P\}$ is a singleton, we will simply write $\mathcal{I}_P$ rather than $\mathcal{I}_{\{P\}}$.

It is known that every t-(v, k, λ_t) design is also an s-(v, k, λ_s) design, $0 \leq s < t$, where $\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}$. Applying this for $s = 1$ and $s = 0$ one finds, in particular, that $|\mathcal{I}_P| = \lambda_1$ for every point P, and that $|\mathcal{B}| = \lambda_0$.

In view of the above paragraph λ_s must be an integer for $0 \leq s < t$; these are the trivial necessary conditions for the existence of a t-(v, k, λ_t) design.

Note, in particular, that the parameters 3-$(16, 7, 5)$ satisfy these conditions:

$$\lambda_0 = 80,\ \lambda_1 = 35,\ \lambda_2 = 14. \tag{1}$$

An *automorphism* of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation on $\mathcal{P}$ leaving $\mathcal{B}$ invariant. The set $\mathrm{Aut}\mathcal{D}$ of all automorphisms of $\mathcal{D}$ is a group under composition which is called *the full automorphism group* of $\mathcal{D}$. The group generated by an automorphism α is denoted by $\langle\alpha\rangle$. Obviously, if $\alpha \in \mathrm{Aut}\mathcal{D}$, then $\langle\alpha\rangle \leq \mathrm{Aut}\mathcal{D}$.

For an automorphism $\alpha \in \mathrm{Aut}\mathcal{D}$, we denote by $\mathrm{fix}(\alpha)$ the set of points of $\mathcal{D}$ fixed by α and, similarly, by $\mathrm{Fix}(\alpha)$ the set of blocks of $\mathcal{D}$ fixed by α. Throughout this article, we shall refer to the orbits of $\mathcal{P}$ or $\mathcal{B}$ under G as the point orbits or block orbits of $\mathcal{D}$ under G, respectively.

A *decomposition* of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair of partitions

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m$$

$$\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$$

of the point set and the block set, respectively. The decomposition is said to be *tactical* if there exist nonnegative integers ρ_{ij} and κ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, such that each point of $\mathcal{P}_i$ lies in precisely ρ_{ij} blocks of $\mathcal{B}_j$, and each block of $\mathcal{B}_j$ contains precisely κ_{ij} points from $\mathcal{P}_i$. The matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ are the corresponding *tactical decomposition matrices*.

There are two trivial examples of tactical decompositions; one is obtained by putting $n = m = 1$, and the other one by partitioning both $\mathcal{P}$ and $\mathcal{B}$ into singletons.

A non-trivial tactical decomposition of $\mathcal{D}$ can be obtained by considering the action of an automorphism group of $\mathcal{D}$ on $\mathcal{D}$. For further reading on the subject of t-designs and automorphism groups we refer the reader to [1], [2], [4], [6]. Here we give well-known properties that shall be used extensively in our arguments.

Theorem 2.1. *Let G be an automorphism group of a design $\mathcal{D}$. Then the point orbits of $\mathcal{D}$ under G and the block orbits of $\mathcal{D}$ under G form a tactical decomposition of $\mathcal{D}$.*

Lemma 2.2. *Let $G = \langle\alpha\rangle$ be a cyclic automorphism group of a design $\mathcal{D}$, and let*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \ldots \sqcup \mathcal{P}_m,\ \ \mathcal{B} = \mathcal{B}_1 \sqcup \ldots\ \mathcal{B}_n,$$

be the associated tactical decomposition of the orbits of $\mathcal{D}$ under G. Then the following holds.

(a) For each point orbit $\mathcal{P}_i$, the set $\mathcal{I}_{\mathcal{P}_i}$ is a disjoint union of block orbits of $\mathcal{D}$ under G.

(b) Every block $B \in Fix(\alpha)$ is a disjoint union of point orbits of $\mathcal{D}$ under G.

The entries ρ_{ij} and κ_{ij} of the tactical decomposition matrices are related by the formula

$$|\mathcal{P}_i| \cdot \rho_{ij} = |\mathcal{B}_j| \cdot \kappa_{ij} \tag{2}$$

which can be easily obtained by means of a double counting of the size of $\mathcal{P}_i \times \mathcal{B}_j$. These entries also satisfy the system of equations given in the following theorem. We recall the reader that a Stirling number of the second kind is the number of ways to partition a set of n elements into k non-empty subsets.

Theorem 2.3. *[9] Let $(\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with a tactical decomposition*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Let $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_s}$ be mutually distinct with $1 \leq s \leq t$ and let $m_1, \ldots, m_s$ be positive integers such that $m_1 + \cdots + m_s \leq t$. Then the entries of the associated tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ satisfy the following equation in which $\left\{ {u \atop v} \right\}$ denotes a Stirling number of the second kind and $(u)_v := u(u-1)\cdots(u-v+1)$ denotes a falling factorial,

$$\begin{aligned} &\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_1 j}^{m_1 - 1} \kappa_{i_2 j}^{m_2} \cdots \kappa_{i_s j}^{m_s} = \\ &\qquad \sum_{\omega \in \Omega} \lambda_{\omega_1 + \omega_2 + \cdots + \omega_s} \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1 - 1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}, \end{aligned} \tag{3}$$

where

$$\Omega = \{(\omega_1, \ldots, \omega_s) \, : \, 1 \leq \omega_j \leq m_j\}.$$

For 3-designs, the above result can be presented in a form which is much more suitable for computation.

Corollary 2.4. *Let $(\mathcal{P}, \mathcal{B})$ be a 3-(v, k, λ_3) design with a tactical decomposition*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Then the entries of the associated tactical decomposition matrix $\mathcal{K} = [\kappa_{ij}]$ satisfy the following system of equations:

$$\sum_{j=1}^{n} \kappa_{ij} = k,$$

$$\sum_{j=1}^{n} \frac{|\mathcal{B}_j|}{|\mathcal{P}_l|} \kappa_{lj}\kappa_{rj} = \begin{cases} \lambda_2 \cdot |\mathcal{P}_r|, & \text{for } l \neq r, \\ \lambda_1 + \lambda_2 \cdot (|\mathcal{P}_r| - 1), & \text{for } l = r, \end{cases}$$

$$\sum_{j=1}^{n} \frac{|\mathcal{B}_j|}{|\mathcal{P}_l|} \kappa_{lj}\kappa_{rj}\kappa_{sj} =$$

$$= \begin{cases} |\mathcal{P}_r| \cdot |\mathcal{P}_s| \cdot \lambda_3, & \text{for } l \neq r \neq s \neq l, \\ |\mathcal{P}_s| \cdot \lambda_2 + (|\mathcal{P}_r| - 1) \cdot |\mathcal{P}_s| \cdot \lambda_3, & \text{for } l = r \neq s \text{ or } l \neq r = s, \\ \lambda_1 + 3\,(|\mathcal{P}_l| - 1) \cdot \lambda_2 + (|\mathcal{P}_l| - 1) \cdot (|\mathcal{P}_l| - 2) \cdot \lambda_3, & \text{for } l = r = s. \end{cases}$$

3. Tactical decompositions of a 3-(16, 7, 5) design with an automorphism of order 3

From now on, $\mathcal{D}$ will be a putative simple 3-$(16, 7, 5)$ design and τ will be a putative automorphism of $\mathcal{D}$ of order 3. The number of the points and the number of the blocks of $\mathcal{D}$ which are fixed by τ will be denoted by f and F, respectively:

$$f := |\mathrm{fix}(\tau)|, \;\; F := |\mathrm{Fix}(\tau)|.$$

Speaking of a point orbit or of a block orbit, we tacitly mean a point orbit or a block orbit of $\mathcal{D}$ under $\langle\tau\rangle$. Also, speaking of a fixed point or of a fixed block we will always mean a point or a block of $\mathcal{D}$ which is fixed by τ.

By the Orbit-Stabilizer Theorem, the orbits of $\mathcal{D}$ under the action of $\langle\tau\rangle$ have size one or three. Let $\mathcal{O}_1, \ldots, \mathcal{O}_f$ be the point orbits of size one, and let $\mathcal{A}_1, \ldots, \mathcal{A}_F$ be the block orbits of size one. Furthermore, let $\mathcal{P}_{f+1}, \ldots, \mathcal{P}_m$ be

the point orbits of size three, and let $\mathcal{B}_{F+1}, \ldots, \mathcal{B}_n$ be the block orbits of size three. Thus

$$\begin{aligned} \mathcal{P} &= \mathcal{O}_1 \sqcup \ldots \sqcup \mathcal{O}_f \sqcup \mathcal{P}_{f+1} \sqcup \ldots \sqcup \mathcal{P}_m, \\ \mathcal{B} &= \mathcal{A}_1 \sqcup \ldots \sqcup \mathcal{A}_F \sqcup \mathcal{B}_{F+1} \sqcup \ldots \sqcup \mathcal{B}_n \end{aligned} \tag{4}$$

is a tactical decomposition of $\mathcal{D}$.

The index of a point orbit (block orbit) indicates the row (column) of the tactical decomposition matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ associated with the orbit. We illustrate here the general form of $\mathcal{K}$.

	$\mathcal{A}_1 \cdots \mathcal{A}_F$	$\mathcal{B}_{F+1} \cdots \mathcal{B}_n$
$\mathcal{O}_1$ $\vdots$ $\mathcal{O}_f$	$\mathcal{K}^{(a)}$	$\mathcal{K}^{(b)}$
$\mathcal{P}_{f+1}$ $\vdots$ $\mathcal{P}_m$	$\mathcal{K}^{(c)}$	$\mathcal{K}^{(d)}$

(5)

It is straightforward to see that the coefficients κ_{ij} and ρ_{ij} of the tactical decomposition (4) satisfy the following conditions.

Lemma 3.1. *(a) Let $i \leq f$ and $j \leq F$. Then $\kappa_{ij} = 1$ or 0 according to whether $\mathcal{O}_i$ is contained in the only block of $\mathcal{A}_j$ or not, respectively.*

(b) Let $i \leq f$ and $j > F$. Then $\kappa_{ij} = 1$ or 0 according to whether $\mathcal{O}_i$ is contained in every block of $\mathcal{B}_j$ or not, respectively.

(c) Let $i > f$ and $j \leq F$. Then $\kappa_{ij} = 3$ or 0 according to whether $\mathcal{P}_i$ is contained in the only block of $\mathcal{A}_j$ or not, respectively.

(d) Let $i > f$ and $j > F$. Then $\rho_{ij} = \kappa_{ij}$ and one of the following conditions holds:

- *(i) $\rho_{ij} = \kappa_{ij} = 0$ if and only if $B \cap \mathcal{P}_i = \emptyset$ for every $B \in \mathcal{B}_j$.*
- *(ii) $\rho_{ij} = \kappa_{ij} = 1$ if and only if $|B \cap \mathcal{P}_i| = 1$ for every $B \in \mathcal{B}_j$ and each point of $\mathcal{P}_i$ is contained in a unique block of $\mathcal{B}_j$.*
- *(iii) $\rho_{ij} = \kappa_{ij} = 2$ if and only if $|B \cap \mathcal{P}_i| = 2$ for every $B \in \mathcal{B}_j$ and each 2-subset of $\mathcal{P}_i$ is contained in a unique block of $\mathcal{B}_j$.*
- *(iv) $\rho_{ij} = \kappa_{ij} = 3$ if and only if $\mathcal{P}_i \subseteq B$ for every $B \in \mathcal{B}_j$.*

We are going to see that, for $i \leq f$ and $j \leq F$, the coefficients κ_{ij} of (4), namely those appearing in the submatrix $\mathcal{K}^{(a)}$ of (5), satisfy an additional system of equations.

Lemma 3.2. *Let $i, r, s \leq f$ and $i \neq r \neq s \neq i$. Then we have:*

$$\begin{aligned}
&\sum_{j=1}^{F} \kappa_{ij} = |\mathcal{I}_{\mathcal{O}_i} \cap Fix(\tau)|, \\
&\sum_{j=1}^{F} \kappa_{ij}\kappa_{rj} = |\mathcal{I}_{\mathcal{O}_i} \cap \mathcal{I}_{\mathcal{O}_r} \cap Fix(\tau)|, \\
&\sum_{j=1}^{F} \kappa_{ij}\kappa_{rj}\kappa_{sj} = |\mathcal{I}_{\mathcal{O}_i} \cap \mathcal{I}_{\mathcal{O}_r} \cap \mathcal{I}_{\mathcal{O}_s} \cap Fix(\tau)|.
\end{aligned} \tag{6}$$

Proof. Let $P_i \in \mathcal{O}_i, P_r \in \mathcal{O}_r$ and $P_s \in \mathcal{O}_s$. The first equation follows straightforward from Lemma 3.1.

The double counting of the set

$$\{(P_r, B) \,:\, \{P_i, P_r\} \subseteq B, B \in \mathrm{Fix}(\tau)\}$$

yields

$$\sum_{j=1}^{F} \rho_{ij}\kappa_{rj} = |\mathcal{I}_{\mathcal{O}_i} \cap \mathcal{I}_{\mathcal{O}_r} \cap \mathrm{Fix}(\tau)|.$$

In this case, by (2), we have $\rho_{ij} = \kappa_{ij}$ and hence we get the second equation.

Finally, the double counting of the set

$$\{(P_r, P_s, B) \,:\, \{P_i, P_r, P_s\} \subseteq B, B \in \mathrm{Fix}(\tau)\}$$

yields the last equation. □

We shall often use the following statements.

Lemma 3.3. *(a) If $\mathcal{S}$ is a set of $\mu \leq 3$ fixed points, then the number of fixed blocks containing $\mathcal{S}$ is congruent to two modulo three:*

$$|\mathcal{I}_{\mathcal{S}} \cap Fix(\tau)| \equiv 2 \pmod 3.$$

(b) The number of fixed blocks containing a given point orbit $\mathcal{P}_i$ of size three is congruent to two modulo three:

$$|\mathcal{I}_{\mathcal{P}_i} \cap Fix(\tau)| \equiv 2 \pmod 3.$$

Proof. (a) Let $\mathcal{S} \subseteq \mathrm{fix}(\tau)$ with $|\mathcal{S}| = \mu \leq 3$. Then $|\mathcal{I}_\mathcal{S}| = \lambda_\mu$ and $\mathcal{I}_\mathcal{S}$ is a disjoint union of block orbits (Lemma 2.2). The block orbits of $\mathcal{D}$ have size one and three, therefore

$$|\mathcal{I}_\mathcal{S} \cap \mathrm{Fix}(\tau)| \equiv \lambda_\mu \pmod 3$$

and the assertion then follows from (1).

(b) A point orbit $\mathcal{P}_i$ has size three, hence $|\mathcal{I}_{\mathcal{P}_i}| = \lambda_3$. Applying the same arguments as in the proof of (a) we obtain

$$|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| \equiv \lambda_3 \pmod 3.$$

□

More precisely, Lemma 3.3 (b) states that a point orbit $\mathcal{P}_i$ is contained in two or five fixed blocks:

$$|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| \in \{2, 5\}.$$

In Lemma 3.1 we determined the admissible entries of the matrix $\mathcal{K} = [\kappa_{ij}]$. We are now especially interested in Lemma 3.1 (d) describing the submatrix $\mathcal{K}^{(d)}$ represented in (5). The admissible values of the coefficients κ_{ij} of this submatrix belong to the set $\{0, 1, 2, 3\}$. Furthermore, we have $\kappa_{ij} = \rho_{ij}$. In order to investigate the multiplicity of each admissible value in $\mathcal{K}^{(d)}$, we need to introduce the coefficients γ_i^s defined as follows:

$$\gamma_i^s := |\{\mathcal{B}_j \,:\, \rho_{ij} = s\}|, \;\; f + 1 \leq i \leq m; \; 0 \leq s \leq 3. \tag{7}$$

Lemma 3.4. *For a point orbit $\mathcal{P}_i$, one of the following statements is valid:*

(a) $|\mathcal{I}_{\mathcal{P}_i} \cap \mathit{Fix}(\tau)| = 2$ *and* $(\gamma_i^1, \gamma_i^2, \gamma_i^3) = (12, 9, 1)$*;*

(b) $|\mathcal{I}_{\mathcal{P}_i} \cap \mathit{Fix}(\tau)| = 5$ *and* $(\gamma_i^1, \gamma_i^2, \gamma_i^3) = (12, 9, 0)$.

Proof. By definition, the coefficient γ_i^s is the number of $\mathcal{B}_j$s such that $|\mathcal{I}_P \cap \mathcal{B}_j| = s$ for each $P \in \mathcal{P}_i$. Hence, by Lemma 3.1 (d), we have

$$\gamma_i^2 = \lambda_2 - \lambda_3 = 9 \text{ and } \gamma_i^1 = \lambda_1 - 2\gamma_i^2 - \lambda_3 = 12.$$

Note that $\rho_{ij} = 3$ if and only if every block of $\mathcal{B}_j$ contains $\mathcal{P}_i$. Every point orbit $\mathcal{P}_i$ of size three is contained in $\lambda_3 = 5$ blocks of $\mathcal{D}$. By Lemma 3.3, $\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)$ has size five or two. In the former case we have $\gamma_i^3 = 0$. In the latter case $\mathcal{P}_i$ is contained in precisely three non-fixed blocks B_1, B_2, B_3. Then, by Lemma 2.2, $\{B_1, B_2, B_3\}$ is a block orbit of $\mathcal{D}$ and $\gamma_i^3 = 1$. □

In the following lemma we determine the admissible values for F which we recall is the number of blocks of $\mathcal{D}$ fixed by τ.

Lemma 3.5. *We have* $F \equiv 2 \pmod 3$ *and* $F \leq 17$.

Proof. Applying Lemma 3.3 (a) with $\mathcal{S} = \emptyset$ we get $F \equiv 2 \pmod 3$.

For a point $P \in \mathcal{P}_i$ we denote by Ω_P the set of block orbits of $\mathcal{D}$ which are not disjoint with $\mathcal{I}_P$:

$$\Omega_P = \{\mathcal{B}_j \,:\, \mathcal{I}_P \cap \mathcal{B}_j \neq \emptyset\}.$$

The upper bound on $|\Omega_P|$ can be easily computed: $|\Omega_P| \leq (\lambda_0 - F)/3$. Therefore,

$$F \leq \lambda_0 - 3|\Omega_P|. \tag{8}$$

Note that $\mathcal{B}_j \in \Omega_P$ if and only if $\rho_{ij} > 0$. Hence, $|\Omega_P| = \gamma_i^1 + \gamma_i^2 + \gamma_i^3$ and $|\Omega_P| \in \{21, 22\}$ by Lemma 3.4. Hence, by (8), we certainly have $F \leq 80 - 3 \cdot 21 = 17$ and the assertion follows. □

Now we introduce additional parameters that are related to the action of τ on $\mathcal{D}$ and then we will determine a system of equations that these parameters must satisfy. We shall systematically use Lemma 2.2, Lemma 3.3 and Lemma 3.5.

A point orbit $\mathcal{P}_i$ is contained in two or five fixed blocks of $\mathcal{D}$. For $s \in \{2, 5\}$, we denote by o_s the number of point orbits of size three contained in exactly s fixed blocks:

$$o_s := |\{\mathcal{P}_i \,:\, |\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| = s\}|, \;\; s \in \{2, 5\}.$$

Considering that $\mathcal{D}$ has 16 points, we can write

$$16 = f + 3(o_2 + o_5). \tag{9}$$

Now, for each non-negative integer s, denote by f_s the number of fixed points contained in exactly s fixed blocks,

$$f_s := |\{P \in \mathrm{fix}(\tau) \,:\, |\mathcal{I}_P \cap \mathrm{Fix}(\tau)| = s\}|.$$

We already know that if P is a fixed point, then $|\mathcal{I}_P \cap \mathrm{Fix}(\tau)| \equiv 2 \pmod 3$ so that we have $f_s = 0$ for $s \not\equiv 2 \pmod 3$. In view of Lemma 3.5 it is also obvious that $f_s = 0$ for $s > 17$, hence we have:

$$f_2 + f_5 + f_8 + f_{11} + f_{14} + f_{17} = f. \tag{10}$$

A fixed block B is a disjoint union of point orbits. Therefore B has exactly 1, 4 or 7 fixed points. We denote by F_1, F_4 and F_7 the number of fixed blocks of each type:

$$F_s := |\{B \in \mathrm{Fix}(\tau) \,:\, |B \cap \mathrm{fix}(\tau)| = s\}|, \quad s \in \{1, 4, 7\}.$$

Obviously,

$$F_1 + F_4 + F_7 = F. \tag{11}$$

Lemma 3.6. *The parameters f, f_2, f_5, f_8, f_{11}, f_{14}, f_{17}, F, F_1, F_4, F_7, o_2 and o_5 satisfy the following system of equations:*

$$\begin{aligned}
&2f_2 + 5f_5 + 8f_8 + 11f_{11} + 14f_{14} + 3(5o_5 + 2o_2) = 7F;\\
&F_4 + 2F_1 = 5o_5 + 2o_2;\\
&F_1 + 4F_4 + 7F_7 = 2f_2 + 5f_5 + 8f_8 + 11f_{11} + 14f_{14} + 17f_{17};\\
&16 = f + 3(o_2 + o_5);\\
&F_1 + F_4 + F_7 = F;\\
&f_2 + f_5 + f_8 + f_{11} + f_{14} + f_{17} = f.
\end{aligned} \tag{12}$$

Proof. The first three equations are obtained by means of a double counting of each of the following sets:

(i) $\{(P, B) \,:\, P \in B, B \in \mathrm{Fix}(\tau)\}$;

(ii) $\{(\mathcal{P}_i, B) \,:\, \mathcal{P}_i \subseteq B, B \in \mathrm{Fix}(\tau)\}$;

(iii) $\{(P, B) \,:\, P \in \mathrm{fix}(\tau), P \in B, B \in \mathrm{Fix}(\tau)\}$.

The remaining equations have been already obtained before. □

4. Main theorem

We are now ready to prove our main result. The strategy will be to use the ingredients obtained in the previous section in order to show that f, the number of points fixed by τ, cannot belong to the set $\{1, 4, 7, 10, 13\}$. On the other hand, by (9), we should have $f \equiv 1 \pmod 3$. So the only possibility would be $f = 16$ but, in this case, τ would be the identity map contradicting the fact that its order is 3 and then our main non-existence result will follow.

4.1. One fixed point

In this subsection we prove that τ cannot fix precisely one point of $\mathcal{D}$.

Lemma 4.1. *Assume that $f = 1$, namely that τ fixes precisely one point of $\mathcal{D}$. Then*

$$F_4 = F_7 = 0, F_1 = F, f_F = 1 \text{ and } f_s = 0 \text{ for } s \neq F.$$

Furthermore, one of the following conditions holds:

(a) $F = 5$ and $o_2 = 5$;

(b) $F = 8$ and $o_2 = 3$.

Proof. By assumption $\mathcal{D}$ has exactly one fixed point and hence exactly five point orbits of size three. Thus we can write

$$\mathcal{P} = \mathcal{O}_1 \sqcup \mathcal{P}_2 \sqcup \ldots \sqcup \mathcal{P}_6.$$

A fixed block B is a disjoint union of point orbits and hence there is a suitable pair (i, j) such that

$$B = \mathcal{O}_1 \sqcup \mathcal{P}_i \sqcup \mathcal{P}_j,$$

and

$$F_4 = 0, F_7 = 0, F = F_1 \leq \binom{5}{2}.$$

Furthermore, the unique fixed point is contained in every fixed block,

$$f_F = 1 \text{ and } f_s = 0, \text{ for } s \neq F.$$

The system of equations (12) is now:

$$2F = 25 - 3o_2. \tag{13}$$

By Lemma 3.5, $F \equiv 2 \pmod 3$. In addition $\mathcal{D}$ has five point orbits of size three so that $o_2 \leq 5$. Therefore $(F, o_2) \in \{2, 5, 8\} \times \{0, 1, 2, 3, 4, 5\}$ and then, by (13), one can see that (a) and (b) are the only possible cases. □

Lemma 4.2. $f \neq 1$.

Proof. Assume that $f = 1$. By Lemma 4.1, we have to distinguish the two cases $(F, o_2) = (5, 5)$ and $(F, o_2) = (8, 3)$.

1st case: $F = o_2 = 5$. The associated tactical decomposition matrix $\mathcal{K} = [\kappa_{ij}]$ has dimension 6×30. A fixed block B is a disjoint union of point orbits, $B = \mathcal{O}_1 \sqcup \mathcal{P}_{i_1} \sqcup \mathcal{P}_{i_2}$. Furthermore, $o_2 = 5$ implies that every point orbit $\mathcal{P}_i$ is contained in exactly two fixed blocks. In addition, the coefficients κ_{ij} satisfy the assertions of Lemma 3.1. Hence, without loss of generality, $\mathcal{K}$ has the following form.

$$\mathcal{K} = \left[\begin{array}{c|c} 1\,1\,1\,1\,1 & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ \hline 3\,3\,0\,0\,0 & \\ 3\,0\,3\,0\,0 & \\ 0\,3\,0\,3\,0 & \mathcal{K}^{(d)} \\ 0\,0\,3\,0\,3 & \\ 0\,0\,0\,3\,3 & \end{array}\right]$$

By Lemma 3.1 (d) we have $\kappa_{ij} = \rho_{ij} \in \{0, 1, 2, 3\}$ for every entry κ_{ij} of the submatrix $\mathcal{K}^{(d)}$. Also, by Lemma 3.4, the multiplicity of 1, 2 and 3 in any row of $\mathcal{K}^{(d)}$ is given by 12, 9 and 1, respectively. Thus every row of $\mathcal{K}^{(d)}$ is a rearrangement of

$$(3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0).$$

Now, let's try to build the submatrix $\mathcal{K}^{(d)}$, row by row, taking into account that the coefficients κ_{ij} of $\mathcal{K}$ must satisfy the equations of Corollary 2.4. For this, we use the program *orbmat6qd* written by V. Krčadinac [8]. We first discover that for the first row of $\mathcal{K}^{(d)}$ there are only two possibilities that are

$$(311111111222222222111000) \quad \text{or} \quad (222111110322222211111100).$$

Then, for each of the above instances, we find that for the second row of $\mathcal{K}^{(d)}$ there are also only two possibilities that are

$$(022211111222211111300222) \quad \text{or} \quad (022211111222211110210321)$$

in the former case, and

$$(110221111212211103222102 1) \quad \text{or} \quad (110221111202222112211103 1)$$

in the latter. At this point, once that the first and second row of $\mathcal{K}^{(d)}$ have been fixed, we see that there is no possible choice for the third one. Hence we have got a contradiction.

2nd case: $F = 8$ and $o_2 = 3$. Here we have $o_5 = 2$, i.e., $\mathcal{D}$ has two point orbits of size three each of which is contained in exactly five fixed blocks. Let $\mathcal{P}_i$ be a point orbit such that $|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| = 5$. A fixed block is a disjoint union of fixed orbits. Therefore, any two blocks of $\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)$ share precisely four points which are the unique fixed point and the points of $\mathcal{P}_i$. Hence, $\mathcal{D}$ has at least $|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| \cdot 3 + |\mathcal{P}_i| + f = 5 \cdot 3 + 3 + 1 = 19$ points; a contradiction.

Both cases lead us to a contradiction and therefore $f \neq 1$. □

4.2. Four fixed points

Here we prove that τ cannot fix precisely four points.

Lemma 4.3. *If $f = 4$, then the following conditions hold:*

(a) $o_2 \leq 4$;

(b) $F_7 = 0$;

(c) $F_4 = 2$;

(d) $F_1 \leq 9$;

(e) $F \leq 11$;

(f) $2f_2 + 5f_5 + 8f_8 + 11f_{11} + 14f_{14} + 17f_{17} \leq 17$;

(g) $f_{11} = f_{14} = f_{17} = 0$.

Proof. (a) We have exactly four point orbits of size three, therefore $o_2 \leq 4$.

(b) The block-size of $\mathcal{D}$, that is $k = 7$, exceeds the number of fixed points of $\mathcal{D}$, therefore $F_7 = 0$.

(c) A fixed block contains one or four fixed points. By Lemma 3.3 the number of fixed blocks containing any set of three fixed points is congruent to 2 (mod 3). Thus we have:

$$F_4 = |\mathcal{I}_{\mathrm{fix}(\tau)} \cap \mathrm{Fix}(\tau)| \in \{2, 5\}.$$

Assume that $F_4 = 5$. If $B \in \mathcal{I}_{\mathrm{fix}(\tau)} \cap \mathrm{Fix}(\tau)$, then $B = \mathrm{fix}(\tau) \sqcup \mathcal{P}_i$ for a suitable i. Thus the intersection of any two fixed blocks of $\mathcal{I}_{\mathrm{fix}(\tau)}$ is $\mathrm{fix}(\tau)$ since $\mathcal{D}$ is simple by assumption. This would imply that $\mathcal{D}$ has at least $|\mathcal{I}_{\mathrm{fix}(\tau)} \cap \mathrm{Fix}(\tau)| \cdot 3 + f = 19$ points; a contradiction. Hence, $F_4 = 2$.

(d) A point orbit of size three is contained in λ_3 blocks, therefore

$$2F_1 + F_4 \leq \frac{16-f}{3}\lambda_3.$$

Thus, having $\lambda_3 = 5$, $f = 4$, and $F_4 = 2$, we obtain the assertion.

Taking into account (b), (c) and (d), the assertions (e) and (f) are immediately obtainable from the equations

$$F = F_1 + F_4 + F_7,$$

$$2f_2 + 5f_5 + 8f_8 + 11f_{11} + 14f_{14} + 17f_{17} = F_1 + 4F_4 + 7F_7$$

of Lemma 3.6.

(g) Let $P \in \mathrm{fix}(\tau)$ and let $B \in \mathcal{I}_P \cap \mathrm{Fix}(\tau)$. Then

$$B = \{P\} \sqcup \mathcal{P}_{i_1} \sqcup \mathcal{P}_{i_2} \quad \text{or} \quad B = \mathrm{fix}(\tau) \sqcup \mathcal{P}_i.$$

Therefore,

$$|\mathcal{I}_P \cap \mathrm{Fix}(\tau)| \leq \binom{4}{2} + F_4 = 8.$$

This means that a fixed point is contained in at most eight fixed blocks, i.e., (g) holds. □

Lemma 4.4. $f \neq 4$.

Proof. Assume that $f = 4$. One can see that the 8-tuples of parameters (F, F_1, F_4, f_2, f_5, f_8, o_2, o_8) satisfying both Lemma 3.6 and Lemma 4.3 are those represented in the following table.

Table 1: The admissible parameters for $f = 4$.

	F	F_1	F_4	f_2	f_5	f_8	o_2	o_5
1.	5	3	2	3	1	0	4	0
2.	8	6	2	2	2	0	2	2
3.	8	6	2	3	0	1	2	2
4.	11	9	2	1	3	0	0	4
5.	11	9	2	2	1	1	0	4

We are going to show that each of these 8-tuples leads to a contradiction.

1st case: $(F, F_1, F_4, f_2, f_5, f_8, o_2, o_5) = (5, 3, 2, 3, 1, 0, 4, 0)$.

Here $\mathcal{D}$ has exactly five fixed blocks, two fixed blocks containing $\mathrm{fix}(\tau)$, and a unique fixed point contained in every fixed block. Furthermore every point orbit of size three is contained in precisely two fixed blocks.

We distinguish two cases according to whether there exists a non-fixed block containing $\mathrm{fix}(\tau)$. Without loss of generality, in the affirmative case the matrix $\mathcal{K}$ has the form

$$\mathcal{K} = \left[\begin{array}{c|c} 11111 & 11111111110000000000000000 \\ 11000 & 11110000001111111100000000 \\ 11000 & 10001110001110000111100000 \\ 11000 & 10000001110001110111010000 \\ \hline 30300 & \multirow{4}{*}{$\mathcal{K}^{(d)}$} \\ 03030 & \\ 00303 & \\ 00033 & \end{array}\right]$$

while, in the negative case, $\mathcal{K}$ has the form

$$\mathcal{K} = \left[\begin{array}{c|c} 11111 & 11111111110000000000000000 \\ 11000 & 11110000001111111100000000 \\ 11000 & 10001110001110000111100000 \\ 11000 & 01001001101001100110011000 \\ \hline 30300 & \multirow{4}{*}{$\mathcal{K}^{(d)}$} \\ 03030 & \\ 00303 & \\ 00033 & \end{array}\right].$$

In both cases, applying the same arguments as in the proof of Lemmma 4.2, one can see that every row of $\mathcal{K}^{(d)}$ is a rearrangement of

$$(3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0).$$

On the other hand, taking into account the restrictions given by Corollary 2.4, we have checked by computer that none of the two above given matrices can be completed with rearrangements of this tuple.

2nd case: $(F, F_1, F_4, f_2, f_5, f_8, o_2, o_5) = (8, 6, 2, 2, 2, 0, 2, 2)$.

Here our approach is the same as in the first case. Without loss of generality we have

$$\mathcal{K} = \left[\begin{array}{c|c} 11111000 & 111111111100000000000000 \\ 11000111 & 111100000011111100000000 \\ 11000000 & 100011100011100011110000 \\ 11000000 & 100000011100011111101000 \\ \hline \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \\ & \end{array}\right]$$

when there exists a non-fixed block containing $\mathrm{fix}(\tau)$, while we have

$$\mathcal{K} = \left[\begin{array}{c|c} 11111000 & 111111111100000000000000 \\ 11000111 & 111100000011111100000000 \\ 11000000 & 100011100011100011110000 \\ 11000000 & 010010011010011011001100 \\ \hline \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \\ & \end{array}\right]$$

when $\mathrm{fix}(\tau)$ is not contained in any non-fixed block.

In both cases we have $o_2 = o_5 = 2$ by assumption. Thus, without loss of generality, we can put

$$|\mathcal{I}_{\mathcal{P}_5} \cap \mathrm{Fix}(\tau)| = |\mathcal{I}_{\mathcal{P}_6} \cap \mathrm{Fix}(\tau)| = 5 \text{ and}$$

$$|\mathcal{I}_{\mathcal{P}_7} \cap \mathrm{Fix}(\tau)| = |\mathcal{I}_{\mathcal{P}_8} \cap \mathrm{Fix}(\tau)| = 2.$$

By Lemma 3.1, the first two rows of $\mathcal{K}^{(c)}$ are rearrangements of

$$(3, 3, 3, 3, 3, 0, 0, 0).$$

Also, by Lemma 3.4, we have $(\gamma_1^i, \gamma_2^i, \gamma_3^i) = (12, 9, 0)$ for $i = 5, 6$. Hence the first two rows of $\mathcal{K}^{(d)}$ are rearrangements of

$$(2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0).$$

Analogously, the last two rows of $\mathcal{K}^{(c)}$ are rearrangements of

$$(3, 3, 0, 0, 0, 0, 0, 0)$$

and, considering that for $i = 7, 8$ we have $(\gamma_1^i, \gamma_2^i, \gamma_3^i) = (12, 9, 1)$ by Lemma 3.4, the last two rows of $\mathcal{K}^{(d)}$ are rearrangements of

$$(3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0).$$

On the other hand, taking into account the restrictions given by Corollary 2.4, we have checked by computer that none of the two above given matrices can be completed with rearrangements of this tuples.

3rd case: $(F, F_1, F_4, f_2, f_5, f_8, o_2, o_5) = (8, 6, 2, 3, 0, 1, 2, 2)$.

Here we have eight fixed blocks and four fix points. In addition, there is a fixed point P belonging to all fixed blocks, i.e., $\mathrm{Fix}(\tau) \subseteq \mathcal{I}_P$. Furthermore, $o_5 = 2$ implies that there exists a point orbit $\mathcal{P}_i$ such that $|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| = 5$. Therefore $|B \cap B'| \geq 4$ for every pair of blocks $B, B' \in \mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)$. On the other hand a fixed block is a disjoint union of point orbits so that we necessarily have $|B \cap B'| = 4$. Therefore $\mathcal{D}$ has at least $1 + |\mathcal{P}_i| + |\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| \cdot 3 = 19$ points; a contradiction.

4th case: $(F, F_1, F_4, f_2, f_5, f_8, o_2, o_5) = (11, 9, 2, 1, 3, 0, 0, 4)$.

Here, without loss of generality, we have

$$\mathcal{K} = \left[\begin{array}{c|c} 1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0 & 1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,1 & 1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0 \\ \hline \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \end{array}\right]$$

when there exists a non-fixed block containing $\mathrm{fix}(\tau)$, while we have

$$\mathcal{K} = \left[\begin{array}{c|c} 1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0 & 1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,1 & 1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\,0 \\ \hline \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \end{array}\right]$$

when $\mathrm{fix}(\tau)$ is not contained in any non-fixed block.

For the assumption $o_5 = 4$, we have $|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| = 5$ for every point orbit $\mathcal{P}_i$ of size three. Then one can see that every row of $\mathcal{K}^{(c)}$ is a rearrangement of

$$(3, 3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0).$$

Also, every row of $\mathcal{K}^{(d)}$ is a rearrangement of

$$(2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0).$$

On the other hand, taking into account the restrictions given by Corollary 2.4, we have checked by computer that none of the two above given matrices can be completed with rearrangements of this tuples.

5th case: $(F, F_1, F_4, f_2, f_5, f_8, o_2, o_5) = (11, 9, 2, 2, 1, 1, 0, 4)$.

Here there is no fixed block containing all fixed points. Indeed the existence of such a block would imply the existence of a 2-set $\mathcal{S} \subseteq \mathrm{fix}(\tau)$ such that $|\mathcal{I}_{\mathcal{S}}| < \lambda_2$, which is not possible.

Therefore, without loss of generality, $\mathcal{K}$ has the following form:

$$\mathcal{K} = \left[\begin{array}{c|c} 1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0 & 1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1 & 1\,1\,1\,1\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,1\,1\,1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,1\,0\,0\,1\,1\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0\,1\,1\,0\,0 \\ \hline \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \end{array}\right]$$

By assumption $o_5 = 4$ and for every point orbit $\mathcal{P}_i$ of size three we have $|\mathcal{I}_{\mathcal{P}_i} \cap \mathrm{Fix}(\tau)| = 5$. Hence, every row of $\mathcal{K}^{(c)}$ is a rearrangement of

$$(3, 3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0).$$

Also, every row of $K^{(d)}$ is a rearrangement of

$$(2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0).$$

But also here it is possible to check that no pair of rearrangements of the above tuples gives a 34-tuple which can be the fifth row of $\mathcal{K}$. □

4.3. More than four fixed points

In this subsection we prove that τ cannot fix more than four points.

Lemma 4.5. *If $f \in \{7, 10, 13\}$, then the following conditions hold.*

(a) There exists a set $\mathcal{S}$ of three fixed points such that $|\mathcal{I}_{\mathcal{S}} \cap Fix(\tau)| = 2$.

(b) A set $\mathcal{S}$ of two fixed points is contained in at least five fixed blocks:

$$|\mathcal{I}_{\mathcal{S}} \cap Fix(\tau)| \geq 5.$$

(c) A fixed point P belongs to at least eight fixed blocks:

$$|\mathcal{I}_{P} \cap Fix(\tau)| \geq 8.$$

Proof. (a) We assume the opposite: every set of three fixed points is contained in λ_3 fixed blocks. Then,

$$\binom{f}{3}\lambda_3 \leq \binom{7}{3}F.$$

Introducing $f = 10$ or $f = 13$ into the above given expression we obtain that $F > 17$ contradicting Lemma 3.5.

If $f = 7$, then $F_7 \in \{0, 1\}$. The double counting of the set

$$\{(\mathcal{S}', B) \,:\, \mathcal{S}' \subseteq \mathrm{fix}(\tau), |\mathcal{S}'| = 3, B \in \mathcal{I}_{\mathcal{S}'} \cap \mathrm{Fix}(\tau)\}$$

yields

$$\binom{f}{3}\lambda_3 = \binom{4}{3}F_4 + \binom{7}{3}F_7.$$

Introducing the admissible values of F_7 into the above given expression, we obtain a contradiction.

(b) Assume the opposite: there exists a 2-subset $\mathcal{S} \subseteq \mathrm{fix}(\tau)$ such that $|\mathcal{I}_{\mathcal{S}} \cap \mathrm{Fix}(\tau)| < 5$. Let $P \in \mathrm{fix}(\tau) \setminus \mathcal{S}$ and $\mathcal{S}' = \mathcal{S} \sqcup \{P\}$. Obviously,

$$\mathcal{I}_{\mathcal{S}'} \cap \mathrm{Fix}(\tau) \subseteq \mathcal{I}_{\mathcal{S}} \cap \mathrm{Fix}(\tau).$$

By Lemma 3.3, if $\mathcal{S}$ is a set of μ fixed points, $1 \leq \mu \leq 3$, then the number of fixed blocks containing $\mathcal{S}$ is congruent to two modulo three. Therefore,

$$|\mathcal{I}_{\mathcal{S}} \cap \mathrm{Fix}(\tau)| = 2 \text{ and } \mathcal{I}_{\mathcal{S}'} \cap \mathrm{Fix}(\tau) = \mathcal{I}_{\mathcal{S}} \cap \mathrm{Fix}(\tau).$$

This implies that

$$\mathcal{I}_{\mathcal{S}} \cap \mathrm{Fix}(\tau) = \bigcap_{P \in \mathrm{fix}(\tau)} \mathcal{I}_P \cap \mathrm{Fix}(\tau) = \mathcal{I}_{\mathrm{fix}(\tau)} \cap \mathrm{Fix}(\tau).$$

For $f \geq 7$ this is not possible since $\mathcal{D}$ is simple, and each block of $\mathcal{D}$ has size seven.

(c) Assume the opposite: there exists a point $P \in \mathrm{fix}(\tau)$ such that $|\mathcal{I}_P \cap \mathrm{Fix}(\tau)| < 8$. Then, $|\mathcal{I}_P \cap \mathrm{Fix}(\tau)| \in \{2, 5\}$. Applying the same arguments as in the proof of (b) we obtain that $\mathrm{fix}(\tau) \subseteq B$ for every $B \in \mathcal{I}_P \cap \mathrm{Fix}(\tau)$, which is not possible. □

Lemma 4.6. *$f < 7$.*

Proof. Let $f \in \{7, 10, 13\}$. By Lemma 4.5 (a), there exists a 3-set $\mathcal{S}$ of fixed points such that $|\mathcal{I}_S \cap \mathrm{Fix}(\tau)| = 2$. Also, by Lemma 4.5, every set of two fixed points is contained in at least five fixed blocks. Therefore, without loss of generality, the tactical decomposition matrix $\mathcal{K} = [\kappa_{ij}]$ has the following form:

$$\mathcal{K} = \left[\begin{array}{l|l} 1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\ldots & \\ 1\,1\,1\,1\,1\,0\,0\,0\,1\,1\,1\ldots & \\ 1\,1\,0\,0\,0\,1\,1\,1\,1\,1\,1\ldots & \mathcal{K}^{(b)} \\ \vdots & \\ \hline & \\ \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \\ & \end{array}\right]$$

Furthermore, the coefficients of matrix $\mathcal{K}$ satisfy the system of equations from Lemma 3.2. Introducing the results of Lemma 3.3 and Lemma 4.5 into these equations we obtain that each solution has the following common part of $\mathcal{K}$.

$$\mathcal{K} = \left[\begin{array}{l|l} 1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\ldots & \\ 1\,1\,1\,1\,1\,0\,0\,0\,1\,1\,1\ldots & \\ 1\,1\,0\,0\,0\,1\,1\,1\,1\,1\,1\ldots & \mathcal{K}^{(b)} \\ 1\,1\ldots & \\ \vdots\,\vdots & \\ 1\,1\ldots & \\ \hline & \\ \mathcal{K}^{(c)} & \mathcal{K}^{(d)} \\ & \end{array}\right]$$

This implies that there exist at least two fixed blocks of $\mathcal{D}$ containing all fixed points of $\mathcal{D}$, which is not possible for $f \geq 7$. □

4.4. Conclusion

We are finally able to prove our main result.

Theorem 4.7. *A simple* 3-$(16, 7, 5)$ *design with an automorphism of order* 3 *does not exist.*

Proof. On one hand we should have $f \in \{1, 4, 7, 10, 13\}$; on the other hand, as shown in the previous subsections, this is not possible. The assertion follows. □

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