ON THE EXTENDABILITY OF PARTICULAR CLASSES OF CONSTANT DIMENSION CODES

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Abstract. In classical coding theory, different types of extendability results of codes are known. A classical example is the result stating that every $(4, q^2-1, 3)$-code over an alphabet of order q is extendable to a $(4, q^2, 3)$-code. A constant dimension subspace code is a set of $(k-1)$-spaces in the projective space $\mathrm{PG}(n-1, q)$, which pairwise intersect in subspaces of dimension upper bounded by a specific parameter. The theoretical upper bound on the sizes of these constant dimension subspace codes is given by the Johnson bound. This Johnson bound relies on the upper bound on the size of partial s-spreads, i.e., sets of pairwise disjoint s-spaces, in a projective space $\mathrm{PG}(N, q)$. When $N+1 \equiv 0 \pmod{s+1}$, it is possible to partition $\mathrm{PG}(N, q)$ into s-spaces, also called s-spreads of $\mathrm{PG}(N, q)$. In the finite geometry research, extendability results on large partial s-spreads to s-spreads in $\mathrm{PG}(N, q)$ are known when $N+1 \equiv 0 \pmod{s+1}$. This motivates the study to determine similar extendability results on constant dimension subspace codes whose size is very close to the Johnson bound. By developing geometrical arguments, avoiding having to rely on extendability results on partial s-spreads, such extendability results for constant dimension subspace codes are presented.

1. Introduction

Presently, a new direction in coding theory, called random network coding, receives a lot of attention. In random network coding, information is transmitted through a network whose topology can vary. A classical example is a wireless network where users come and go. For more details see [1, 19, 21, 22, 27]. R. Kötter and F. Kschischang [21] proved in an inspiring article that a very good way of transmission is obtained in networks if *subspace codes* are used. Here, the codewords are k-dimensional vector subspaces of the n-dimensional vector space

1991 *Mathematics Subject Classification.* 05B25, 51E20.

Key words and phrases. Random network coding; extendability of codes; minihypers.

$\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ of order q. To transmit a codeword, i.e. a k-dimensional vector space, through the network, it is sufficient to transmit a basis of this k-dimensional vector space. But a k-dimensional subspace has different bases. Kötter and Kschischang proved that the transmission can be optimized if the nodes in the network transmit linear combinations of the incoming basis vectors of the k-dimensional subspace which represents the codeword. These ideas led to many new interesting problems in coding theory, in Galois geometries and in design theory. For instance, it leads to the study of q-analogues of Steiner systems. A *Steiner system* $S_q(t,k,n)$ is a set of k-dimensional subspaces of $\mathbb{F}_q^n$, where each t-dimensional subspace is contained in precisely one k-dimensional subspace of $S_q(t,k,n)$. It is known that the $S_q(1,k,n)$ exists if and only if k divides n. However, for $t \geq 2$ there is only few known examples: they are Steiner systems $S_2(2,3,13)$ [5]. Arguably the most important open problem in this field is the question of the existence of a Steiner system $S_q(7,3,1)$, known as a q-analogue of the Fano plane (see [6, 11, 16, 20, 23, 24, 25, 26]). For other open problems, see [10].

In classical coding theory, a *q-ary code* is a set of sequences of symbols from an alphabet F of order q. A *q-ary (n,M,d)-code* is a code containing M codewords each having length n, and having minimum distance d. In classical coding theory, extendability results are known. The example we give relies on a result on Latin squares. A *Latin square* of order q is a $q \times q$ array whose entries form a set F of q distinct symbols such that each row and each column of the array contain each symbol exactly once. Two Latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ of order q are said to be *mutually orthogonal* if the q^2 pairs (a_{ij}, b_{ij}) are all distinct. A classical example of extendability result states that every $(4, q^2-1, 3)$-code over an alphabet of order q is extendable to a $(4, q^2, 3)$-code. A particular application is the result that every 6-ary $(4,35,3)$-code $\mathcal{C}$ is extendable to a 6-ary $(4,36,3)$-code $\mathcal{C}'$ [17]. But since a $(4,36,3)$-code $\mathcal{C}'$ is equivalent to the existence of two mutually orthogonal Latin squares of order 6, which do not exist, this extendability result implies that no 6-ary $(4,35,3)$-code $\mathcal{C}$ exists. As there is an example of a 6-ary $(4,34,3)$-code, this leads to the known value $A_6(4,3) = 34$ [17].

We present in this paper a similar extendability result on specific constant dimension subspace codes. An (n,M,d,k)-subspace code over $\mathbb{F}_q$ is a set of M k-dimensional subspaces of $\mathbb{F}_q^n$, having minimum distance d.

We first concentrate on large $(n, M, 2k-2, k)$-subspace codes over $\mathbb{F}_q$. The upper bound on the maximal size $M = \mathcal{A}_q(n, 2k-2, k)$ of an $(n, M, 2k-2, k)$-code over $\mathbb{F}_q$ is

$$M \leq \begin{bmatrix} n \\ 2 \end{bmatrix}_q \Big/ \begin{bmatrix} k \\ 2 \end{bmatrix}_q . \tag{1}$$

Assume that $n \equiv 0 \pmod{k}$ and $n-1 \equiv 0 \pmod{k-1}$, so that this upper bound $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right]_q$ is an integer. The main question is whether this upper bound is sharp. But assume that $M = \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right]_q - \delta$ for some small positive integer δ. Here, in our arguments, $\delta \leq (q+1)/2$ needs to be assumed. Then by using the theory of minihypers in finite projective spaces, it is possible to prove that this $(n, M, 2k-2, k)$-code can be extended to an $(n, \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right]_q, 2k-2, k)$-code. This means that if no $(n, \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right]_q, 2k-2, k)$-code exists, the upper bound (1) can be improved by $(q+1)/2$.

This extendability result is then generalized to a more general extendability result on other classes of constant dimension random network codes, whose parameters satisfy specific divisibility conditions.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field of order q and let $\mathbb{F}_q^n$ be the vector space of dimension n over $\mathbb{F}_q$. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

The number of k-dimensional subspaces containing a fixed t-dimensional subspace, $t \leq k$, is

$$\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q .$$

The *subspace distance* of subspaces V and W of $\mathbb{F}_q^n$ is defined by

$$d(U, V) = \dim(U+V) - \dim(U \cap V).$$

A *subspace code* $\mathcal{C}$ is a non-empty collection of subspaces of $\mathbb{F}_q^n$. A *codeword* is an element of $\mathcal{C}$. The *minimum distance of* $\mathcal{C}$ is defined by

$$d(\mathcal{C}) := \min\{d(U,V) \,|\, U, V \in \mathcal{C}, U \neq V\}.$$

We say that a code $\mathcal{C}$ is an (n, M, d)*-subspace code* if $\mathcal{C}$ is of cardinality M and has minimum distance d. A *constant dimension subspace code* is a code with all codewords of the same dimension. If all codewords of an (n, M, d)-code $\mathcal{C}$ are k-dimensional subspaces of $\mathbb{F}_q^n$ and we say that

$\mathcal{C}$ is an (n, M, d, k)-*subspace code*. The maximal number of codewords in an (n, M, d, k)-code is denoted by $\mathcal{A}_q(n, d, k)$.

2.1. **Codes in projective spaces.** We will investigate codes in the projective setting.

The *projective space* $\mathrm{PG}(n, q)$ of projective dimension n over $\mathbb{F}_q$ is the lattice of subspaces of $\mathbb{F}_q^{n+1}$ with respect to set inclusion. In this article, a *k-space*, $0 \leq k \leq n$, is a subspace $\mathrm{PG}(k, q)$ of $\mathrm{PG}(n, q)$ of projective dimension k. The number of projective points of a k-space is $\begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$. A *hyperplane* of $\mathrm{PG}(n, q)$ is an $(n-1)$-space. Alternatively, $\mathrm{PG}(n, q)$ can be defined as point-line geometry [8]. Let Δ be a k-space of $\mathrm{PG}(n, q)$. If subspaces of $\mathrm{PG}(n, q)$ of dimension $k+1$ and $k+2$ containing Δ are considered as new *points* and *lines*, with inclusion as incidence, then they correspond to projective geometry $\mathrm{PG}(n-k)$ called the *quotient geometry of* Δ. We shall denote the quotient geometry of Δ as $\mathrm{PG}(n-k)_\Delta$. An m-space of $\mathrm{PG}(n, q)$ that contains Δ has projective dimension $m - k$ in $\mathrm{PG}(n-k)_\Delta$.

We will use some of the well-known combinatorial structures defined in finite projective spaces. More precisely, we will use a particular type of blocking sets, known under the name of minihypers. We will also refer to results on partial spreads in finite projective spaces. Here, we will mention in particular extendability results of partial spreads to larger (partial) spreads. These results were obtained by using connections with blocking sets and minihypers. It were precisely these extendability results on partial spreads that inspired the research described in this article.

To situate the results on the combinatorial structures that we will apply, we first describe these results in the next paragraphs.

A *blocking set* in $\mathrm{PG}(2, q)$ is a set of points intersecting every line of $\mathrm{PG}(2, q)$. We say that a blocking set B is *trivial* if B contains a line, and B is *minimal* if no proper subset of B still is a blocking set.

The following results on the smallest non-trivial blocking sets of $\mathrm{PG}(2, q)$ are known.

Theorem 2.1. *Let B be a non-trivial blocking set of the smallest possible size in $\mathrm{PG}(2, q)$, and let $|B| = q + \epsilon$.*

1. *If $2 < q$ is a prime, then $\epsilon = (q + 3)/2$ [3].*
2. *If q is a square, then $\epsilon = \sqrt{q} + 1$ [7].*
3. *If q is a non-square, $q = p^h$, $h > 2$, p prime, then $\epsilon = q^{2/3} + 1$ for $p > 3$ and $\epsilon = q^{2/3}/2^{1/3} + 1$ for $p = 2, 3$ [4].*

Blocking sets are closely related to other finite structures in projective geometries. Here we introduce a blocking multiset: a *minihyper*. Blocking sets in $\mathrm{PG}(2,q)$ were used to obtain characterization results on minihypers, and we shall use these properties in the continuation of this article.

Definition 2.2. [15] *A weighted* $\{f,m;n,q\}$-minihyper *is a pair* (F,w), *where* F *is a subset of the point set of* $\mathrm{PG}(n,q)$ *and where* w *is a weight function* $w:\mathrm{PG}(n,q)\to\mathbb{N}$, *satisfying:*

(1) $w(P)>0 \Leftrightarrow P\in F$,
(2) $\sum_{P\in F} w(P)=f$, *and,*
(3) $\min\{\sum_{P\in H} w(P)\,|\,H$ *is a hyperplane*$\}=m$.

A *sum* of k-subspaces of $\mathrm{PG}(n,q)$ is a weight function from the set of k-spaces of $\mathrm{PG}(n,q)$ onto $\mathbb{N}$. Such a sum induces a weight function on subspaces of smaller dimension. In particular, the weight of a point P is the sum of the weights of the k-spaces passing through P. The following theorems present some results on minihypers.

Theorem 2.3. [13] *Let* (F,w) *be a* $\{\sum_{i=0}^{n-1}\epsilon_i\begin{bmatrix}i+1\\1\end{bmatrix}_q,\sum_{i=1}^{n-1}\epsilon_i\begin{bmatrix}i\\1\end{bmatrix}_q;n,q\}$-*minihyper, where* $0\le\epsilon_i\le q-1$, $i=0,\dots,n-1$, *then:*

(1) *If* m *is an integer such that* $1\le m\le n$, *then* $|(F,w)\cap\Omega|\ge\sum_{i=m}^{n-1}\epsilon_i\begin{bmatrix}i-m+1\\1\end{bmatrix}_q$ *for any* $(n-m)$-*space* Ω *in* $\mathrm{PG}(n,q)$ *and the equality is valid for some* $(n-m)$-*space* Ω *in* $\mathrm{PG}(n,q)$.
(2) $|(F,w)\cap\Delta|\ge\sum_{i=2}^{n-1}\epsilon_i\begin{bmatrix}i-1\\1\end{bmatrix}_q$ *for any* $(n-2)$-*space* Δ *in* $\mathrm{PG}(n,q)$ *and* $|(F,w)\cap G|=\sum_{i=2}^{n-1}\epsilon_i\begin{bmatrix}i-1\\1\end{bmatrix}_q$ *for some* $(n-2)$-*space* G *in* $\mathrm{PG}(n,q)$.

Let H_j, $j=1,\dots,q+1$, *be the* $q+1$ *hyperplanes in* $\mathrm{PG}(n,q)$ *through an* $(n-2)$-*space* G *intersecting* F *in* $\sum_{i=2}^{n-1}\epsilon_i\begin{bmatrix}i-1\\1\end{bmatrix}_q$ *points. Then* $(F,w)\cap H_j$ *is a*

$$\{\delta_j+\sum_{i=1}^{n-1}\epsilon_i\begin{bmatrix}i\\1\end{bmatrix}_q,\sum_{i=2}^{n-1}\epsilon_i\begin{bmatrix}i-1\\1\end{bmatrix}_q;n-1,q\}\text{-minihyper}$$

in H_j, *for* $j=1,\dots,q+1$, *and the parameters* δ_j *are some non-negative integers such that* $\sum_{j=1}^{q+1}\delta_j=\epsilon_0$.

Theorem 2.4. [14] *Let* G *be any* $(n-2)$-*space in* $\mathrm{PG}(n,q)$ *and let* $H_0,\dots,H_q$ *be the* $q+1$ *hyperplanes in* $\mathrm{PG}(n,q)$ *that contain* G. *Let* A_i *be a* $(\lambda-1)$-*space in* H_i $(i=0,\dots,q)$ *such that:*

(a) $G\cap A_0=\dots=G\cap A_q=B$ *for a* $(\lambda-2)$-*space* B *in* G, *and,*
(b) $E_\alpha\cap A_\alpha=B$ *for some integer* α *in* $\{2,\dots,q\}$,

where $2 \leq \lambda < n$ *and* $E_\alpha = H_\alpha \cap \langle A_0, A_1\rangle$.

Let Δ *be an* $(n-3)$*-space in* G *and let* $D_\alpha = G \cap \langle E_\alpha, A_\alpha\rangle$.

Then:

(1) *In the case* $B \subset \Delta$ *and* $D_\alpha \not\subset \Delta$, *let* $\Pi_1, \dots, \Pi_q$ *be the* q *hyperplanes in* $\mathrm{PG}(n,q)$ *different from* H_α *that contain the* $(n-2)$*-space* $\langle \Delta, E_\alpha\rangle$. *Then there exists a hyperplane* Π *in* $\{\Pi_1, \dots, \Pi_q\}$ *such that* $|\Pi \cap (\bigcup_{i=0}^{q} A_i)| = |\mathrm{PG}(\lambda-2,q)|$.

(2) *In the case* $B \not\subset \Delta$, $|\langle \Delta, E_\alpha\rangle \cap (\bigcup_{i=0}^{q} A_i)| = |\mathrm{PG}(\lambda-1,q)|$.

Lemma 2.5. [12] *Let* (F,w) *be a* $\{\delta\begin{bmatrix}\mu+1\\1\end{bmatrix}_q, \delta\begin{bmatrix}\mu\\1\end{bmatrix}_q; n-1, q\}$*-minihyper satisfying* $1 \leq \delta \leq (q+1)/2$, $0 \leq \mu \leq n-2$, *and containing a* μ*-space* Δ. *Then the minihyper* (F', w') *where*

$$w'(P) = \begin{cases} w(P)-1 & , P \in \Delta, \\ w(P) & , P \in \mathrm{PG}(n-1,q) \setminus \Delta, \end{cases}$$

is a $\{(\delta-1)\begin{bmatrix}\mu+1\\1\end{bmatrix}_q, (\delta-1)\begin{bmatrix}\mu\\1\end{bmatrix}_q; n-1, q\}$*-minihyper.*

Theorem 2.6. [12] *Let* $q+\epsilon$ *denote the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$.

If (F,w) *is a* $\{\delta\begin{bmatrix}\mu+1\\1\end{bmatrix}_q, \delta\begin{bmatrix}\mu\\1\end{bmatrix}_q; n-1, q\}$*-minihyper,* $q > 2$, *satisfying* $0 \leq \delta < \epsilon$ *and* $\mu < n-1$, *then* w *is the weight function induced on the points of* $\mathrm{PG}(n-1,q)$ *by a sum of* δ μ*-spaces.*

A $(k-1)$-*spread* of $\mathrm{PG}(n-1,q)$ is a partitioning of the point set of $\mathrm{PG}(n-1,q)$ in $(k-1)$-spaces. A $(k-1)$-spread of $\mathrm{PG}(n-1,q)$ exists if and only if $n \equiv 0 \pmod{k}$. Then, the size of a $(k-1)$-spread is $\begin{bmatrix}n\\1\end{bmatrix}_q / \begin{bmatrix}k\\1\end{bmatrix}_q$.

A *partial* $(k-1)$-*spread* $\mathcal{S}$ of $\mathrm{PG}(n-1,q)$ is a set of pairwise disjoint $(k-1)$-spaces of $\mathrm{PG}(n-1,q)$. We say that $\mathcal{S}$ is *complete* if $\mathcal{S}$ cannot be extended to a larger partial $(k-1)$-spread of $\mathrm{PG}(n-1,q)$. A *hole* is a point of $\mathrm{PG}(n-1,q)$ that is not contained in any $(k-1)$-space of $\mathcal{S}$. Assume that $n \equiv 0 \pmod{k}$. If $\mathcal{S}$ is a partial $(k-1)$-spread and $|\mathcal{S}| = \begin{bmatrix}n\\1\end{bmatrix}_q / \begin{bmatrix}k\\1\end{bmatrix}_q - \delta$, then we say that $\mathcal{S}$ has *deficiency* δ. In [12], it was shown that partial $(k-1)$-spreads are closely related to minihypers.

Theorem 2.7. [12] *Let* $\mathcal{S}$ *be a partial* $(k-1)$*-spread of* $\mathrm{PG}(n-1,q)$, *with* $n \equiv 0 \pmod{k}$ *and with deficiency* $\delta < q$, *and let* F *be the set of holes of* $\mathcal{S}$. *Then* F *is a* $\{\delta\begin{bmatrix}k\\1\end{bmatrix}_q, \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q; n-1, q\}$*-minihyper.*

The previous theorem was then used, in combination with Theorem 2.6, to show that in some cases partial $(k-1)$-spreads are extendable to $(k-1)$-spreads.

Theorem 2.8. [12] *Let $n \equiv 0 \pmod{k}$. Let $q+\epsilon$ be the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*, $q>2$. If $\delta<\epsilon$, then every partial $(k-1)$-spread of* $\mathrm{PG}(n-1,q)$ *of deficiency δ is extendable to a $(k-1)$-spread of* $\mathrm{PG}(n-1,q)$*.*

Partial spreads are closely related to subspace codes. A $(k-1)$-spread of $\mathrm{PG}(n-1,q)$ is a maximal $(n,M,2k,k)$-code of cardinality

$$M=\mathcal{A}_q(n,2k,k)=\begin{bmatrix}n\\1\end{bmatrix}_q\Big/\begin{bmatrix}k\\1\end{bmatrix}_q,\qquad \text{when } n\equiv 0 \pmod{k}.$$

A partial $(k-1)$-spread of $\mathrm{PG}(n-1,q)$, $n\equiv 0 \pmod{k}$, of size M is also an $(n,M,2k,k)$-code, not necessarily of the maximal size. However, by Theorem 2.8, $(n,M,2k,k)$-codes obtained from partial $(k-1)$-spreads of $\mathrm{PG}(n-1,q)$, $n\equiv 0 \pmod{k}$, $M=\mathcal{A}_q(n,2k,k)-\delta=\left[\begin{smallmatrix}n\\1\end{smallmatrix}\right]_q/\left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]_q-\delta$, $\delta>0$, δ small, can be extended to maximal $(n,M'=\left[\begin{smallmatrix}n\\1\end{smallmatrix}\right]_q/\left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]_q,2k,k)$-codes meeting the Johnson bound (Theorem 2.10).

Our main aim is to obtain a similar extendability result for other classes of constant dimension codes. This is achieved in Theorems 3.6 and 4.2.

When k does not divide n, the following lower bound on $\mathcal{A}_q(n,2k,k)$ is known.

Theorem 2.9. [2, 10] *Let $n\equiv r \pmod{k}$, $0\le r\le k-1$. Then for all q, we have*

$$\frac{q^n-q^k(q^r-1)-1}{q^k-1}\le \mathcal{A}_q(n,2k,k).$$

In general, in the terminology of projective geometry, an (n,M,d,k)-code $\mathcal{C}$ can be viewed as a set of $(k-1)$-spaces of $\mathrm{PG}(n-1,q)$ of cardinality M with the following property: $t=k-d/2+1$ is the smallest integer such that every $(t-1)$-space of $\mathrm{PG}(n-1,q)$ is contained in at most one $(k-1)$-space of $\mathcal{C}$. Equivalently, any two codewords of $\mathcal{C}$ intersect in at most a $(t-2)$-space. Hence, $(k-1)$-spaces of $\mathcal{C}$ through a $(t-2)$-space Δ form a partial $(k-t)$-spread $\mathcal{C}_\Delta$ in the quotient geometry $\mathrm{PG}(n-t,q)$ of Δ. Note that $\mathcal{C}_\Delta$ is an $(n-k+d/2,M',d,d/2)$-code called the *quotient code of* Δ. Using this observation, the following theorem was obtained. The upper bound on the size of a constant dimension code, presented in the next theorem, is known under the name *Johnson bound*.

Theorem 2.10 (Johnson bound)**.** [10] *The maximal size* $\mathcal{A}_q(n,d,k)$ *of an* (n,M,d,k)*-code satisfies the upper bound*

$$\mathcal{A}_q(n,d,k) \leq \frac{\begin{bmatrix} n \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} k \\ t-1 \end{bmatrix}_q} \mathcal{A}_q(n-k+d/2, d, d/2),$$

where $t = k - d/2 + 1$ *and where* $\mathcal{A}_q(n-k+d/2, d, d/2)$ *is the maximal size of a partial* $(d/2-1)$*-spread in* $\mathrm{PG}(n-k+d/2-1, q)$*.*

Corollary 2.11. *The maximal size of an* (n,M,d,k)*-code, with* $n - k + d/2 \equiv 0 \pmod{d/2}$*, satisfies the upper bound*

$$\mathcal{A}_q(n,d,k) \leq \frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}.$$

The main problem is whether there exist constant dimension codes meeting the Johnson bound.

Here, Honold, Kiermaier and Kurz proved that $\mathcal{A}_2(6,4,3) = 77$ [18].

We will prove an extendability result for constant dimension codes whose size is very close to the Johnson bound. We will prove this in two steps. We first of all present an extendability result on a particular class of $(n, M, 2k-2, k)$-codes.

3. Extendability of a particular class of $(n, M, 2k-2, k)$-codes

In this section, we shall assume that $n \equiv 0 \pmod{k}$ and $(n-1) \equiv 0 \pmod{k-1}$. Let $\mathcal{C}$ be an $(n, M, 2k-2, k)$-code. By Corollary 2.11,

$$|\mathcal{C}| = M \leq \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q.$$

In the projective space setting, the code $\mathcal{C}$ is a set of $(k-1)$-spaces in $\mathrm{PG}(n-1,q)$ intersecting pairwise in at most a point of $\mathrm{PG}(n-1,q)$. We define the *surplus* of a point P of $\mathrm{PG}(n-1,q)$ as the number of $(k-1)$-spaces in $\mathcal{C}$ containing P:

$$sur_{\mathcal{C}}(P) = |\{\Delta \mid \Delta \in \mathcal{C}, P \in \Delta\}|.$$

It is well-known that the $(k-1)$-spaces of the code $\mathcal{C}$, sharing precisely a given point P, form a partial $(k-2)$-spread, of cardinality at most $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$, in the quotient geometry $\mathrm{PG}(n-1,q)_P$ of P. Therefore,

$$sur_{\mathcal{C}}(P) \leq \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q.$$

It is known from Theorem 2.8 that partial $(k-2)$-spreads of $\mathrm{PG}(n-1,q)_P$, $(n-1) \equiv 0 \pmod{k-1}$, of small positive deficiency δ', can be extended via δ' different $(k-2)$-spaces $\Pi_1, \ldots, \Pi_{\delta'}$, to a $(k-2)$-spread of $\mathrm{PG}(n-1,q)_P$. If this is indeed the case for the partial $(k-2)$-spread in the quotient geometry of P, defined by the $(k-1)$-spaces of the code $\mathcal{C}$ through P; suppose it can be extended by $w(P)$ different $(k-2)$-spaces $\Pi_1, \ldots, \Pi_{w(P)}$ to a $(k-2)$-spread in $\mathrm{PG}(n-1,q)_P$, then the $(k-1)$-spaces $\langle P, \Pi_1 \rangle, \ldots, \langle P, \Pi_{w(P)} \rangle$ form obvious candidates for $(k-1)$-spaces extending the $(n, M, 2k-2, k)$-code $\mathcal{C}$, to a larger $(n, M+w(P), 2k-2, k)$-code $\mathcal{C}'$.

This is illustrated by Figure 1. The codewords of $\mathcal{C}$ through this point P are represented by the triangles through P whose sides consist of full lines. Their intersections with the quotient geometry $\mathrm{PG}(n-1,q)_P$ are represented with full lines, so these full lines define a partial $(k-2)$-spread in $\mathrm{PG}(n-1,q)_P$. The dotted lines inside the space $\mathrm{PG}(n-1,q)_P$ represent the $(k-2)$-spaces extending this partial $(k-2)$-spread to a $(k-2)$-spread in $\mathrm{PG}(n-1,q)_P$. By extending these $(k-2)$-spaces with the point P, we obtain the $(k-1)$-spaces through P represented by the triangles whose sides are in dotted lines. They are the obvious candidate $(k-1)$-spaces $\langle P, \Pi_1 \rangle, \ldots, \langle P, \Pi_{w(P)} \rangle$ through P which extend the constant dimension code $\mathcal{C}$ to a larger constant dimension code having the same minimum distance.

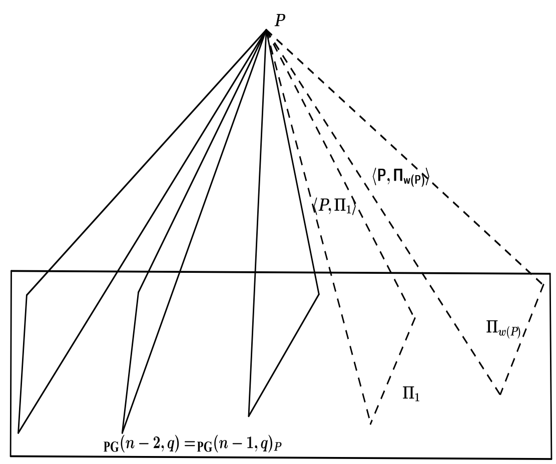

Figure 1: Extendability of partial $(k-2)$-spreads in $\mathrm{PG}(n-2,q)$

The arguments presented below indeed will prove this. Moreover, our developed arguments will prove the extendability of the $(n, M, 2k-$

$2,k)$-code $\mathcal{C}$, avoiding having to rely on the extendability of the partial $(k-2)$-spreads in the quotient geometries $\mathrm{PG}(n-1,q)_P$ of the points P of (F,w).

Lemma 3.1. *Let $n \equiv 0 \pmod{k}$ and $(n-1) \equiv 0 \pmod{k-1}$. Let $\mathcal{C}$ be an $(n, M, 2k-2, k)$-code and let $M = \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q - \delta$, $1 \leq \delta \leq q-1$. Then $\mathcal{C}$ defines a weighted $\{\delta \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \delta \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q; n-1, q\}$-minihyper (F,w), where*

$$F = \left\{ P \in \mathrm{PG}(n-1,q) \,|\, sur_{\mathcal{C}}(P) < \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q \right\}$$

and

$$w(P) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - sur_{\mathcal{C}}(P).$$

Proof. To show that (F,w) is a minihyper, we compute $\sum_{P \in \mathrm{PG}(n-1,q)} w(P)$ and $\sum_{P \in H} w(P)$ for an arbitrary hyperplane H of $\mathrm{PG}(n-1,q)$. The total weight of the minihyper (F,w) equals

$$\begin{aligned} \sum_{P \in \mathrm{PG}(n-1,q)} w(P) &= \sum_{P \in \mathrm{PG}(n-1,q)} \left(\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - sur_{\mathcal{C}}(P) \right) = \\ &= \begin{bmatrix} n \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - |\mathcal{C}| \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \delta \begin{bmatrix} k \\ 1 \end{bmatrix}_q. \end{aligned}$$

A hyperplane H of $\mathrm{PG}(n-1,q)$ either contains a $(k-1)$-space or intersects a $(k-1)$-space in a $(k-2)$-space. Denote by α the number of $(k-1)$-spaces of $\mathcal{S}$ contained in H. Then

$$\begin{aligned} \sum_{P \in H} w(P) &= \sum_{P \in H} \left(\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - sur_{\mathcal{C}}(P) \right) \\ &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - \left(|\mathcal{C}| \cdot \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q + \alpha q^{k-1} \right) \\ &= q^{k-1} \left(\begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q - \alpha \right) - q^{n-1} \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q + \delta \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q \\ &\equiv \delta \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q \pmod{q^{k-1}}. \end{aligned}$$

Hence, (F,w) is a weighted $\{\delta \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \delta \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q; n-1, q\}$-minihyper. □

Lemma 3.2. *Let $n \equiv 0 \pmod{k}$, $(n-1) \equiv 0 \pmod{k-1}$ and $1 \leq \delta \leq (q+1)/2$. Let $\mathcal{C}$ be an $(n, M, 2k-2, k)$-code and let $M = \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q - \delta$.*

Consider the $\{\delta \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \delta \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q; n-1, q\}$-minihyper (F,w) defined by the code $\mathcal{C}$.

Consider an $(n-k)$-space Δ such that $|\Delta \cap (F,w)| = \delta$. Then every $(n-k+1)$-space through Δ intersects (F,w) in a sum of δ lines.

Moreover, every point P of (F,w) belongs to $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ lines completely contained in (F,w).

Proof. Let $P \in (F,w)$ be a point of weight $w(P)$. The $(k-1)$-spaces of the code $\mathcal{C}$ through P form a partial spread $\mathcal{S}$ of $(k-2)$-spaces of deficiency $w(P)$ in the quotient geometry $\mathrm{PG}(n-1,q)_P$ of P. Hence, $\mathcal{S}$ has $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ holes in $\mathrm{PG}(n-1,q)_P$. Let R be a hole of $\mathcal{S}$. Let T be a point on the line PR.

If T does not belong to the minihyper (F,w), then $w(T)=0$. Then the $(k-1)$-spaces of $\mathcal{C}$ through T form a $(k-2)$-spread in the quotient geometry $\mathrm{PG}(n-1,q)_T$ of T. However, there is no $(k-1)$-space of $\mathcal{C}$ through the line PR, hence we obtain a contradiction.

Therefore $w(T)>0$, and the line PR lies completely in (F,w). This implies that P belongs to $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ lines of $\mathrm{PG}(n-1,q)$ that are completely contained in (F,w). Hence, the points of $\Delta \cap (F,w)$ belong to, in total, $\delta\begin{bmatrix}k-1\\1\end{bmatrix}_q$ lines contained in (F,w).

By Theorem 2.3, we know that $(n-k+1)$-spaces through Δ intersect (F,w) in $\{\delta\begin{bmatrix}2\\1\end{bmatrix}_q, \delta\begin{bmatrix}1\\1\end{bmatrix}_q; n-k+1, q\}$-minihypers. So, every such $(n-k+1)$-space contains at most δ lines of (F,w).

There are $\begin{bmatrix}k-1\\1\end{bmatrix}_q$ different $(n-k+1)$-spaces through Δ. Assume they contain x_i lines of (F,w), then $x_i \le \delta$. Therefore,

$$\sum_i x_i \quad \le \quad \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q.$$

On the other hand, every line of (F,w) through a point of $\Delta \cap (F,w)$ belongs to one such $(n-k+1)$-space through Δ. Therefore,

$$\delta\begin{bmatrix}k-1\\1\end{bmatrix}_q \quad \le \quad \sum_i x_i \quad \le \quad \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q.$$

Hence, $x_i=\delta$, for every i, and every $(n-k+1)$-space through Δ contains a sum of δ lines of (F,w). □

The next Figure 2 represents the situation we know at this point in the arguments. The partial $(k-2)$-spread in $\mathrm{PG}(n-1,q)_P$ defined by the codewords of $\mathcal{C}$, passing through P, is a partial $(k-2)$-spread in $\mathrm{PG}(n-1,q)_p$ of deficiency $w(P)$.This is again described via the triangles in full lines and their intersections with the quotient geometry $\mathrm{PG}(n-2,q)=\mathrm{PG}(n-1,q)_P$ in Figure 2. We do not know whether this partial $(k-2)$-spread can be extended to a $(k-2)$-spread in $\mathrm{PG}(n-1,q)_P$, but this will not be required. The crucial element in our further arguments is that the point P belongs to $w(P)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines completely consisting of elements of (F, w).

In Figure 2, the holes of the partial $(k-2)$-spread in $\mathrm{PG}(n-1, q)_P$ of deficiency $w(P)$, defined by the codewords of $\mathcal{C}$ through P, are described by the dotted ellipse. The dotted lines in Figure 2 represent the $w(P)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines through P completely consisting of elements of (F, w).

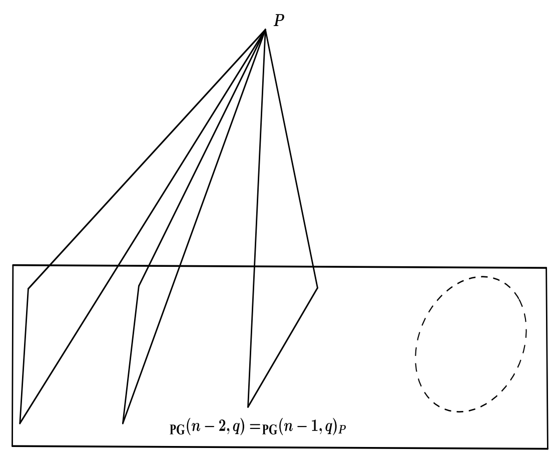

Figure 2: Point P belongs to $w(P)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines contained in (F, w)

Theorem 3.3. *Let* $n \equiv 0 \pmod{k}$, $(n-1) \equiv 0 \pmod{k-1}$ *and* $1 \leq \delta \leq (q+1)/2$. *Let* $\mathcal{C}$ *be an* $(n, M, 2k-2, k)$*-code and let* $M = \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q - \delta$.

Then the $\{\delta\begin{bmatrix} k \\ 1 \end{bmatrix}_q, \delta\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q; n-1, q\}$*-minihyper* (F, w) *defined by* $\mathcal{C}$ *is a sum of* δ $(k-1)$*-spaces.*

Proof. The proof of this theorem is equal to the proof of Theorem 2.6, but that proof relied on links with blocking sets in the projective plane $\mathrm{PG}(2, q)$.

Here, we will rely in the proofs on the fact that every point P of (F, w) belongs to $w(P)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines completely contained in the minihyper (F, w) (Lemma 3.2).

To make the article self-contained and to describe how the arguments of the proof of Theorem 2.6 are developed in the new setting, we present the proof by giving it via the next two lemmas. In this way, we can always assume the upper bound $\delta \leq (q+1)/2$. □

Lemma 3.4. *Let* (F,w) *be a weighted* $\{\delta\begin{bmatrix}3\\1\end{bmatrix}_q, \delta\begin{bmatrix}2\\1\end{bmatrix}_q; n-1, q\}$*-minihyper satisfying the property that every point* P *of* (F,w) *belongs to* $w(P)\begin{bmatrix}2\\1\end{bmatrix}_q$ *lines completely contained in* (F,w)*.*

Then (F,w) *is a sum of* δ *planes of* $\mathrm{PG}(n-1,q)$*.*

Proof. Let (F,w) be a $\{\delta\begin{bmatrix}3\\1\end{bmatrix}_q, \delta\begin{bmatrix}2\\1\end{bmatrix}_q; n-1, q\}$-minihyper satisfying $1 \leq \delta \leq (q+1)/2$.

By Theorem 2.3, there exists an $(n-3)$-space G in $\mathrm{PG}(n-1,q)$ such that $|(F,w)\cap G| = \delta$. Let $H_0, \ldots, H_q$ be the $q+1$ hyperplanes through G. Then $(F,w)\cap H_i$ is a $\{\delta\begin{bmatrix}2\\1\end{bmatrix}_q, \delta\begin{bmatrix}1\\1\end{bmatrix}_q; n-2, q\}$-minihyper with weights in H_i. By Lemma 3.2, $(F,w)\cap H_i$ is a sum of δ lines.

Let P be a point of (F,w) with minimal weight. If $P \notin G$, then we will choose a new $(n-3)$-space G and new hyperplanes $H_0, \ldots, H_q$, to make sure that $P \in G$. To see this, suppose $P \in H_0\backslash G$. We know that $H_0 \cap (F,w)$ is a sum of δ lines. Since there are $\begin{bmatrix}n-2\\1\end{bmatrix}_q$ (resp. $\begin{bmatrix}n-3\\1\end{bmatrix}_q$, $\begin{bmatrix}n-4\\1\end{bmatrix}_q$) $(n-3)$-spaces in H_0 through P (resp. through a line in H_0 containing P, through P and a line in H_0 not containing P), and since $\delta \leq (q+1)/2$, there exists an $(n-3)$-space G_0 through P in H_0 that intersects each of the δ lines in exactly a point. Thus G_0 contains δ points of (F,w).

So we can suppose $P \in G$. We know that $(F,w)\cap H_i$ is a sum of δ lines $A_{i1}, \ldots, A_{i\delta}$ (Lemma 3.2), and that this sum is unique. Moreover, each one of these lines intersects G in a point, such that $G\cap(F,w)$ is a sum of δ points $B_1, B_2, \ldots, B_\delta$, which is also unique.

Case 1.: *Assume the point* P *has weight one.* There is exactly one point in $G\cap(F,w)$ equal to P, e.g. B_1, and exactly one line in $H_i\cap(F,w)$ through P, e.g. A_{i1}. Now suppose the lines $A_{01}, A_{11}, \ldots, A_{q1}$ do not form a plane through P, i.e., suppose $\bigcup_{i=0}^{q} A_{i1} \neq \langle A_{01}, A_{11}\rangle$.

Let $E_i = H_i \cap \langle A_{01}, A_{11}\rangle$. Then there exists an integer $\alpha \in \{2, \ldots, q\}$ such that $E_\alpha \neq A_{\alpha 1}$. Let $D_\alpha = G\cap\langle E_\alpha, A_{\alpha 1}\rangle$, thus D_α is a line in G containing B_1. We know that B_j $(j \in \{2, \ldots, \delta\})$ is different from B_1 since $w(B_1) = w(P) = 1$, thus $\langle B_j, B_1\rangle$ is a line.

Since the number of $(n-4)$-spaces in G through B_1 (resp. through a line in G, through a subspace of dimension greater than one) equals $\begin{bmatrix}n-3\\1\end{bmatrix}_q$ (resp. equals $\begin{bmatrix}n-4\\1\end{bmatrix}_q$, is at most $\begin{bmatrix}n-5\\1\end{bmatrix}_q$), and since $\delta\begin{bmatrix}n-4\\1\end{bmatrix}_q < \begin{bmatrix}n-3\\1\end{bmatrix}_q$, there exists an $(n-4)$-space Δ in G through $P = B_1$, with $B_2 \not\subset \Delta, \ldots, B_\delta \not\subset \Delta$ and $D_\alpha \not\subset \Delta$.

By Theorem 2.4, there exists a hyperplane π through $\langle \Delta, E_\alpha \rangle$ such that $|\pi \cap (\bigcup_{i=0}^{q} A_{i1})| = 1$.

Now consider the lines A_{ij}, where $i > 1$. They intersect G in one of the points B_i, $i > 1$. Since $\pi \cap G = \Delta$, we conclude that $\pi \cap A_{ij}$, $i > 1$, is a point. We now count the points of (F, w) in π, obtaining at most $1 + (\delta - 1)(q + 1) < \delta(q + 1)$ points, a contradiction.

We conclude that the lines $A_{01}, \ldots, A_{q1}$ form a plane through P, contained in (F, w).

Case 2.: *Assume the point P has weight $k > 1$.* In H_i, $i = 0, \ldots, q$, all lines have weight zero or at least k. Indeed, suppose this is not the case, suppose there is a line π in H_0 with positive weight at most $k-1$. Since each point in this line has weight at least k, we need more than $q > \delta$ other lines in H_0 with weight greater than zero to cover the points of π. This is impossible.

In H_i, there are exactly k lines through P. Since the weight of a point of (F, w) is at least k, these k lines must in fact be one line A_{i1} counted k times. The other lines in H_i intersecting G do not pass through P.

Repeating the arguments from Case 1, we conclude that the k-fold lines A_{i1} form a k-fold plane through P, contained in (F, w).

In both cases there exists a plane of (F, w) through P. By Lemma 2.5, this plane can be removed, resulting in a $\{(\delta - 1)\begin{bmatrix}3\\1\end{bmatrix}_q, (\delta - 1)\begin{bmatrix}2\\1\end{bmatrix}_q; n - 1, q\}$-minihyper (F', w').

Repeating this technique for downsizing the minihyper, we obtain the desired result: (F, w) is a sum of δ planes. □

We now prove the general characterization result on the weighted $\{\delta\begin{bmatrix}k\\1\end{bmatrix}_q, \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q; n-1, q\}$-minihyper (F, w) required to prove the extendability result of the constant dimension codes $\mathcal{C}$. This will be proven by induction on k. Here, we rely on the following induction hypotheses:

- $(i = 0)$ if an $(n - k)$-space Δ contains δ points of (F, w), then every $(n-k+1)$-space through Δ contains $\delta\begin{bmatrix}2\\1\end{bmatrix}_q$ points of (F, w), and intersects (F, w) in a $\{\delta\begin{bmatrix}2\\1\end{bmatrix}_q, \delta\begin{bmatrix}1\\1\end{bmatrix}_q; n - k + 1, q\}$-minihyper which is a sum of δ lines;
- $(i = 1)$ if an $(n - k + 1)$-space through Δ has $\delta\begin{bmatrix}2\\1\end{bmatrix}_q$ points of (F, w), then every $(n-k+2)$-space through Δ intersects (F, w) in a $\{\delta\begin{bmatrix}3\\1\end{bmatrix}_q, \delta\begin{bmatrix}2\\1\end{bmatrix}_q; n - k + 2, q\}$-minihyper, which is a sum of δ planes, $\ldots$,

- (for general i) if an $(n-k+i)$-space through Δ has $\delta\begin{bmatrix}i+1\\1\end{bmatrix}_q$ points of (F,w) forming a $\{\delta\begin{bmatrix}i+1\\1\end{bmatrix}_q, \delta\begin{bmatrix}i\\1\end{bmatrix}_q; n-k+i, q\}$-minihyper, then every $(n-k+i+1)$-space through Δ intersects (F,w) in a $\{\delta\begin{bmatrix}i+2\\1\end{bmatrix}_q, \delta\begin{bmatrix}i+1\\1\end{bmatrix}_q; n-k+i+1, q\}$-minihyper forming a sum of δ $(i+1)$-spaces.

Then we can show the following result.

Theorem 3.5. *Let* (F,w) *be a weighted* $\{\delta\begin{bmatrix}k\\1\end{bmatrix}_q, \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q; n-1, q\}$*-minihyper such that every point* P *of* (F,w) *belongs to* $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ *lines completely consisting of elements of* (F,w)*, then* (F,w) *is a sum of* δ $(k-1)$*-spaces.*

Proof. This can again be proven using the techniques of the proof of Theorem 2.6. The cases $i=0$ and $i=1$ of the induction hypothesis have been proven in Lemma 3.2 and Lemma 3.4. □

This now concludes the proof of Theorem 3.3.

We now prove one of the main results of this article. This is the first of the two extendability results of particular constant dimension codes to larger constant dimension codes meeting the Johnson bound.

Theorem 3.6. *Let* $n \equiv 0 \pmod{k}$*,* $(n-1) \equiv 0 \pmod{k-1}$ *and* $1 \leq \delta \leq (q+1)/2$*. Let* $\mathcal{C}$ *be an* $(n, M, 2k-2, k)$*-code, with* $M = \begin{bmatrix}n\\2\end{bmatrix}_q / \begin{bmatrix}k\\2\end{bmatrix}_q - \delta$*. Then* $\mathcal{C}$ *can be extended to an* $(n, M' = \begin{bmatrix}n\\2\end{bmatrix}_q / \begin{bmatrix}k\\2\end{bmatrix}_q, 2k-2, k)$*-code* $\mathcal{C}'$ *meeting the Johnson bound.*

Proof. Consider the $\{\delta\begin{bmatrix}k\\1\end{bmatrix}_q, \delta\begin{bmatrix}k-1\\1\end{bmatrix}_q; n-1, q\}$-minihyper (F,w) defined by the code $\mathcal{C}$. Let $P \in (F,w)$ be a point of weight $w(P)$. The $(k-1)$-spaces of the code $\mathcal{C}$ through P form a partial spread $\mathcal{S}$ of $(k-2)$-spaces of deficiency $w(P)$ in the quotient geometry $\mathrm{PG}(n-1,q)_P$ of P. Hence, $\mathcal{S}$ has $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ holes in $\mathrm{PG}(n-1,q)_P$. In the proof of Lemma 3.2, we showed that the lines joining P to the holes of $\mathcal{S}$ in $\mathrm{PG}(n-1,q)_P$ are completely contained in (F,w). Hence, P belongs to $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ lines of $\mathrm{PG}(n-1,q)$ that are completely contained in (F,w).

On the other hand, the point P belongs to $w(P)$ $(k-1)$-spaces of (F,w) (Theorems 3.3 and 3.5). By Theorems 3.3 and 3.5, (F,w) is a sum of δ $(k-1)$-spaces and every line through P with all points of positive weight is contained in one of those $(k-1)$-spaces.

Consequently, the $w(P)\begin{bmatrix}k-1\\1\end{bmatrix}_q$ lines joining P with the holes of $\mathcal{S}$ in $\mathrm{PG}(n-1,q)_P$ correspond to the lines through P of the $w(P)$ $(k-1)$-spaces through P of (F,w). This shows that the $w(P)$ $(k-1)$-spaces of (F,w) through P pairwise intersect in only this point P. In addition, this implies that each such $(k-1)$-space shares at most the point P with a $(k-1)$-space of $\mathcal{C}$. Therefore, $\mathcal{C}$ can be extended by these $w(P)$ $(k-1)$-spaces of (F,w) through P to a code $\mathcal{C}'$ of size $M+w(P)$, having the same minimum distance.

Repeating the same arguments for all the points of (F,w), we finally obtain that $\mathcal{C}$ can be extended to an $(n, M' = \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q, 2k-2, k)$-code $\mathcal{C}'$ meeting the Johnson bound (Theorem 2.10). □

A straightforward consequence of Theorem 3.6 is the following corollary on the bound on the size of a particular class of $(n, M, 2k-2, k)$-codes.

Corollary 3.7. *Let* $n \equiv 0 \pmod{k}$ *and* $n-1 \equiv 0 \pmod{k-1}$. *Then, one of the following cases holds:*

(1) $\mathcal{A}_q(n, 2k-2, k) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q$, *or*
(2) $\mathcal{A}_q(n, 2k-2, k) < \begin{bmatrix} n \\ 2 \end{bmatrix}_q / \begin{bmatrix} k \\ 2 \end{bmatrix}_q - (q+1)/2$.

4. Generalisation

In this section, we generalize the results from the previous section. The main result is obtained in Theorem 4.2, where we present our most general extendability result on constant dimension codes.

Let $\mathcal{C}$ be an (n, M, d, k)-code and let $t = k - d/2 + 1$. Then $\mathcal{C}$ is a set of $(k-1)$-spaces in $\mathrm{PG}(n-1, q)$ intersecting in at most a $(t-2)$-space. Assume that $n - i \equiv 0 \pmod{k-i}$, for $i = 0, \ldots, t-1$. Then,

$$|\mathcal{C}| = M \leq \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q.$$

We define the *surplus* in $\mathcal{C}$ of a subspace Δ to be the number of $(k-1)$-spaces in $\mathcal{C}$ containing Δ:

$$sur_{\mathcal{C}}(\Delta) = |\{\Omega \mid \Omega \in \mathcal{C}, \Delta \subseteq \Omega\}|.$$

We are especially interested in the case when Δ is a subspace of dimension $t-2$. The $(k-1)$-spaces of $\mathcal{C}$ passing through Δ form a partial $(k-t)$-spread in the quotient geometry $\mathrm{PG}(n-t, q) = \mathrm{PG}(n-1, q)_\Delta$ of Δ. Therefore,

$$sur_{\mathcal{C}}(\Delta) \leq \begin{bmatrix} n-t+1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}_q.$$

Furthermore, we are interested in the upper bound on the surplus of a point P. The double-counting of the set

$$\{(\Delta, \Omega) \mid P \in \Delta, \Delta \subseteq \Omega, \dim\Delta = t-2, \Omega \in \mathcal{C}\}$$

yields

$$sur_{\mathcal{C}}(P) \cdot \begin{bmatrix} k-1 \\ t-2 \end{bmatrix}_q \leq \begin{bmatrix} n-1 \\ t-2 \end{bmatrix}_q \cdot \begin{bmatrix} n-t+1 \\ 1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}_q,$$

and finally

$$sur_{\mathcal{C}}(P) \leq \begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q.$$

Lemma 4.1. *Let $\mathcal{C}$ be an (n, M, d, k)-code, with $t = k - d/2 + 1$.*

Let $n-i \equiv 0 \pmod{k-i}$, for $i = 0, \ldots, t-1$, and let $M = \left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ t \end{smallmatrix}\right]_q - \delta$, $1 \leq \delta \leq q-1$.

Then $\mathcal{C}$ defines a weighted $\{\delta \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q, \delta \left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]_q; n-1, q\}$-minihyper (F, w), where

$$F = \left\{ P \in \mathrm{PG}(n-1, q) \mid sur_{\mathcal{C}}(P) < \begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q \right\}$$

and

$$w(P) = \begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q \Big/ \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q - sur_{\mathcal{C}}(P).$$

Proof. The total weight of the minihyper (F, w) equals

$$\begin{aligned} \sum_{P \in \mathrm{PG}(n-1,q)} w(P) &= \sum_{P \in \mathrm{PG}(n-1,q)} \left(\left[\begin{smallmatrix} n-1 \\ t-1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k-1 \\ t-1 \end{smallmatrix}\right]_q - sur_{\mathcal{C}}(P) \right) = \\ &= \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q \cdot \left[\begin{smallmatrix} n-1 \\ t-1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k-1 \\ t-1 \end{smallmatrix}\right]_q - |\mathcal{S}| \cdot \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q = \delta \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q. \end{aligned}$$

A hyperplane H of $\mathrm{PG}(n-1, q)$ either contains a $(k-1)$-space or intersects this $(k-1)$-space in a $(k-2)$-space. Denote by α the number of $(k-1)$-spaces of $\mathcal{C}$ contained in H. Then

$$\begin{aligned} \sum_{P \in H} w(P) &= \sum_{P \in H} \left(\left[\begin{smallmatrix} n-1 \\ t-1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k-1 \\ t-1 \end{smallmatrix}\right]_q - sur_{\mathcal{C}}(P) \right) \\ &= \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q \cdot \left[\begin{smallmatrix} n-1 \\ t-1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k-1 \\ t-1 \end{smallmatrix}\right]_q - \left(|\mathcal{C}| \cdot \left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]_q + \alpha q^{k-1} \right) \\ &= q^{k-1} \left(\left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k \\ t \end{smallmatrix}\right]_q - \alpha \right) - q^{n-1} \cdot \left[\begin{smallmatrix} n-1 \\ t-1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} k-1 \\ t-1 \end{smallmatrix}\right]_q + \delta \left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]_q \\ &\equiv \delta \left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]_q \pmod{q^{k-1}}. \end{aligned}$$

Hence, (F, w) is a weighted $\{\delta \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q, \delta \left[\begin{smallmatrix} k-1 \\ 1 \end{smallmatrix}\right]_q; n-1, q\}$-minihyper. □

To conclude this article, we present the generalisation of Theorem 2.8. This is the main extendability result of this article.

Theorem 4.2. *Let $\mathcal{C}$ be an (n, M, d, k)-code, with $t = k - d/2 + 1$.*

Assume that $n - i \equiv 0 \pmod{k - i}$, for $i = 0, \ldots, t-1$, and that $M = {n \brack t}_q / {k \brack t}_q - \delta$.

If $\delta \leq (q+1)/2$, then $\mathcal{C}$ can be extended to an $(n, M' = {n \brack t}_q / {k \brack t}_q, d, k)$-code $\mathcal{C}'$ meeting the Johnson bound.

Proof. By Lemma 4.1, $\mathcal{C}$ defines a weighted $\{\delta {k \brack 1}_q, \delta {k-1 \brack 1}_q; n-1, q\}$-minihyper (F, w), of points having positive surplus.

We shall prove that $\mathcal{C}$ can be extended by the δ $(k-1)$-spaces of the minihyper (F, w) to a larger $(n, {n \brack t}_q / {k \brack t}_q, d, k)$-code $\mathcal{C}'$. We prove this by induction on t. For $t = 2$, $\mathcal{C}$ is an $(n, M, 2k-2, k)$-code, and the result follows from Theorem 3.6.

Let $t > 2$. Let P be a point of the minihyper (F, w). Then $0 < w(P) \leq \delta$ and the $(k-1)$-spaces of $\mathcal{C}$ through a point P define a quotient $(n-1, M', d, k-1)$-code $\mathcal{C}_P$ of size $M' = {n-1 \brack t-1}_q / {k-1 \brack t-1}_q - w(P)$. This code $\mathcal{C}_P$ is a set of $(k-2)$-spaces in the quotient geometry $\mathrm{PG}(n-2, q) = \mathrm{PG}(n-1, q)_P$ of P intersecting pairwise in at most a $(t-3)$-space. By the induction hypothesis on $t-1$, since $w(P) \leq (q+1)/2$, the quotient code $\mathcal{C}_P$ defines a weighted $\{w(P) {k-1 \brack 1}_q, w(P) {k-2 \brack 1}_q; n-2, q\}$-minihyper (F_P, w_P) in the quotient geometry $\mathrm{PG}(n-1, q)_P$ of P. By the same induction hypothesis on $t-1$, the quotient code $\mathcal{C}_P$ can be extended by $w(P)$ different $(k-2)$-spaces to a constant dimension code meeting the Johnson bound, and these $w(P)$ distinct $(k-2)$-spaces form a sum of $(k-2)$-spaces which is the weighted minihyper (F_P, w_P).

Let Δ be a $(k-2)$-space of the minihyper (F_P, w_P). Therefore, since Δ extends the quotient code $\mathcal{C}_P$, Δ intersects each codeword of $\mathcal{C}_P$ and each $(k-2)$-space of (F_P, w_P) in at most a $(t-3)$-space. Let $\Omega = \langle P, \Delta \rangle$. We shall show that the $(k-1)$-space Ω is a $(k-1)$-space of the minihyper (F, w).

Let $R \in \Omega \setminus \{P\}$ and $R' \in PR \cap \Delta$. Then, R' is a point of the minihyper (F_P, w_P). Therefore, $w_P(R') > 0$ and $sur_{\mathcal{C}_P}(R') < {n-2 \brack t-2}_q / {k-2 \brack t-2}_q$, i.e. R' is not contained in the maximal number of codewords of the quotient code $\mathcal{C}_P$. Then, the line $R'P = RP$ does not lie in the maximal number of codewords of $\mathcal{C}$. Moreover, $w_P(R')$ is the number of $(k-1)$-spaces in $\mathcal{C}$ that are missing through the line PR'. This implies that $w(R) > 0$ and, consequently, R is a point of the minihyper (F, w). Hence, Ω is a $(k-1)$-space of the minihyper (F, w) through the point P.

Obviously, Ω does not belong to the code $\mathcal{C}$, since the $(k-2)$-space Δ is a $(k-2)$-space of the minihyper (F_P, w_P), extending the quotient code $\mathcal{C}_P$. From the construction of Ω, it follows that Ω intersects a codeword of $\mathcal{C}$ in at most a $(t-2)$-space. Also, the same argument implies that two $(k-1)$-spaces, passing through P, of the minihyper (F, w) intersect in at most a $(t-2)$-space.

The next Figure 3 illustrates this result. The dotted squares in the quotient geometry $\mathrm{PG}(n-1,q)_P$ represent the $w(P)$ different $(k-2)$-spaces $\Pi_1, \ldots, \Pi_{w(P)}$ in the quotient geometry $\mathrm{PG}(n-1,q)_P$ extending the quotient code $\mathcal{C}_P$ to a code meeting the Johnson bound. Then the $w(P)$ different $(k-1)$-spaces $\langle P, \Pi_1\rangle, \ldots, \langle P, \Pi_{w(P)}\rangle$ through P are the $w(P)$ different $(k-1)$-spaces through P extending the code $\mathcal{C}$ to a larger code having the same minimum distance.

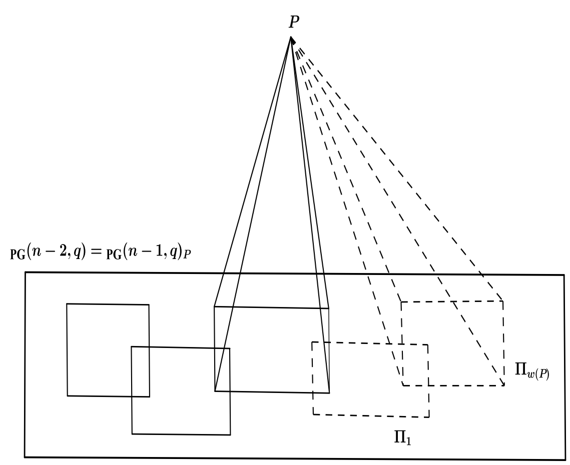

Figure 3: the $w(P)$ different $(k-1)$-spaces through P extending the code $\mathcal{C}$

Repeating this argument, we obtain δ different $(k-1)$-spaces that intersect the codewords of $\mathcal{C}$ in at most a $(t-2)$-space, and that also pairwise intersect in at most a $(t-2)$-space. Therefore, $\mathcal{C}$ can be extended to a constant dimension $(n, \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q, d, k)$-code $\mathcal{C}'$, with $t = k - d/2 + 1$. Equivalently, $\mathcal{C}'$ meets the Johnson bound (Theorem 2.10). □

A straightforward consequence of Theorem 4.2 is the following corollary on the bound on the size of a particular class of (n, M, d, k)-codes.

Corollary 4.3. *Let* $t = k - d/2 + 1$.

Assume that $n - i \equiv 0 \pmod{k-i}$, *for* $i = 0, \ldots, t-1$.

Then, one of the following holds:

(1) $\mathcal{A}_q(n,d,k) = \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q$, *or,*
(2) $\mathcal{A}_q(n,d,k) < \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q - (q+1)/2$.

Acknowledgment
The research of the first author was supported by a Short Term Scientific Mission (STSM) of the COST-project IC-1104 *Random Network Coding and Designs over GF*(q) (`http://www.network-coding.eu/`).

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