Rješenja 3. školske zadaće, grupe 1 i 5

GRUPA A:

1. $D(f) = \langle 0, \infty \rangle$, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \text{L'H} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = 0$, $\lim\limits_{x\to\infty} f(x) = +\infty$.

   $f'(x) = \dfrac{1}{\sqrt{x}}(1 + \dfrac{1}{2}\ln x)$, $f'(x) = 0 \Rightarrow x = e^{-2}$, i to je točka minimuma, tj. f pada na $\langle 0, e^{-2} \rangle$ i raste na $\langle e^{-2}, \infty \rangle$.

   $f''(x) = \dfrac{-\ln x}{4x\sqrt{x}}$, $f''(x) = 0 \Rightarrow x = 1$, i to je točka infleksije, tj. f je konveksna na $\langle 0, 1 \rangle$ i konkavna na $\langle 1, \infty \rangle$.

2.

$$\begin{aligned}
\int e^{\sqrt{x}} dx &= \left\{ \begin{array}{l} t = \sqrt{x} \\ dt = \frac{dx}{2\sqrt{x}} \\ dx = 2tdt \end{array} \right\} = \int e^t 2tdt = \left\{ \begin{array}{l} u = t, du = dt \\ dv = e^t dt \\ v = e^t \end{array} \right\} \\
&= 2te^t - 2\int e^t dt = 2e^t(t-1) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C,\ C \in \mathbb{R}.
\end{aligned}$$

   Napomena: Rješenje neodređenog integrala bez '+C' nosi 2 boda.

3. $\displaystyle\int_0^2 \frac{x+1}{2x^2+1} dx = \int_0^2 \frac{x}{2x^2+1} dx + \int_0^2 \frac{1}{2x^2+1} dx = I_1 + I_2$

   $I_1 = \left\{ \begin{array}{l} t = 2x^2 + 1 \\ dt = 4xdx \\ 0 \to 1, 2 \to 9 \end{array} \right\} = \dfrac{1}{4}\displaystyle\int_1^9 \frac{dt}{t} = \frac{1}{4} \ln t \,\Big|_1^9 = \frac{\ln 9}{4}.$

   $I_2 = \dfrac{1}{2}\displaystyle\int_0^2 \frac{dx}{x^2 + \frac{1}{2}} = \frac{1}{2}\frac{1}{\frac{1}{\sqrt{2}}} \operatorname{arctg}\Big(\frac{x}{\frac{1}{\sqrt{2}}}\Big) \Big|_0^2 = \frac{\sqrt{2}}{2} \operatorname{arctg}(2\sqrt{2}).$

GRUPA B:

1. $D(f) = [0, \infty\rangle$, $\lim_{x \to 0^+} f(x) = f(0) = 0$, $\lim_{x\to\infty} f(x) = \frac{\sqrt{x}}{e^{\sqrt{x}}} = \text{L'H} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}} = 0.$

   $f'(x) = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}(1-\sqrt{x})$, $f'(x) = 0 \Rightarrow x = 1$, i to je točka maksimuma, tj. f raste na $[0,1\rangle$ i pada na $\langle 1, \infty\rangle$.

   $f''(x) = -\frac{e^{-\sqrt{x}}}{4x^{\frac{3}{2}}}(\sqrt{x} - x + 1)$, $f''(x) = 0 \Rightarrow x = \frac{1}{2}(3+\sqrt{5})$, i to je točka infleksije, tj. f je konkavna $[0, \frac{1}{2}(3+\sqrt{5})\rangle$ i konveksna na $\langle \frac{1}{2}(3+\sqrt{5}), \infty\rangle$.

2.

$$\begin{aligned}\int \frac{xdx}{\cos^2 x} &= \left\{ \begin{array}{l} u = x, du = dx \\ dv = \frac{dx}{\cos^2 x} \\ v = \operatorname{tg} x \end{array} \right\} = x \operatorname{tg} x - \int \operatorname{tg} x dx = \left\{ \begin{array}{l} t = \cos x \\ dt = -\sin x dx \end{array} \right\} \\ &= x \operatorname{tg} x + \int \frac{dt}{t} = x \operatorname{tg} x + \ln|t| + C = x \operatorname{tg} x + \ln|\cos x| + C, \;\; C \in \mathbb{R}.\end{aligned}$$

Napomena: Rješenje neodređenog integrala bez '+C' nosi 2 boda.

3. $\int_0^{\frac{1}{2}} \frac{1-x}{\sqrt{1-2x^2}}dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}}dx - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-2x^2}}dx = I_1 - I_2$

   $I_1 = \frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{\frac{1}{2} - x^2}} = \frac{1}{\sqrt{2}} \arcsin(\frac{x}{\frac{1}{\sqrt{2}}}) \,|_0^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \arcsin \frac{\sqrt{2}}{2}.$

   $I_2 = \left\{ \begin{array}{l} t = 1-2x^2 \\ dt = -4xdx \\ 0 \to 1, \frac{1}{2} \to \frac{1}{2} \end{array} \right\} = \frac{-1}{4}\int_1^{\frac{1}{2}} \frac{dt}{\sqrt{t}} = \frac{1}{4}\int_{\frac{1}{2}}^1 \frac{dt}{\sqrt{t}} = \frac{1}{4} 2\sqrt{t}|_{\frac{1}{2}}^1 = \frac{1}{2} - \frac{1}{2\sqrt{2}}.$