# Rješenja 2. školske zadaće, grupe 3, 7 i 9

## Grupa A:

1. *a)*

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n+1}\right)^{n+1}\right]^{\frac{2n}{n+1}} = e^{\lim_{n\to\infty}\frac{2}{1+1/n}} = e^2$$

*b)*

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{2n+1}{n+1}\right)^{\frac{n+1}{n}} &= \left(\lim_{n\to\infty}\frac{2n+1}{n+1}\right)^{\lim_{n\to\infty}\frac{n+1}{n}} \\ &= \left(\lim_{n\to\infty}\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)^{\lim_{n\to\infty}(1+\frac{1}{n})} = 2^1 = 2\end{aligned}$$

2. *i)* Niz $(a_n)$ je padajući niz:
Baza: $a_2 = \frac{10}{3} < 4 = a_1$
Pretpostavka indukcije: $a_n < a_{n-1}$
Korak: $a_n < a_{n-1} \Rightarrow \frac{1}{3}a_n < \frac{1}{3}a_{n-1} \Rightarrow \frac{1}{3}a_n + 2 < \frac{1}{3}a_{n-1} + 2 \Rightarrow a_{n+1} < a_n$

*ii)* Niz $(a_n)$ je ograničen odozdo:
Baza: $a_1 > 3$
Pretpostavka indukcije: $a_n > 3$
Korak: $a_{n+1} = \frac{1}{3}a_n + 2 > \frac{1}{3}\cdot 3 + 2 = 3$

*iii)* Budući da je monoton i ograničen, niz $(a_n)$ je konvergentan i postoji $L = \lim_{n\to\infty} a_n$. Vrijedi:

$$a_{n+1} = \frac{1}{3}a_n + 2 \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{3}a_n + 2\right) \Rightarrow L = \frac{1}{3}L + 2 \Rightarrow L = 3.$$

3. *a)*

$$\begin{aligned}\lim_{x\to+\infty}\frac{3x+1}{\sqrt{x^2+x+1}+x} &= \lim_{x\to+\infty}\frac{3x+1}{\sqrt{x^2+x+1}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \lim_{x\to+\infty}\frac{3+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{3}{2}.\end{aligned}$$

*b)*

$$\begin{aligned}
\lim_{n\to+\infty}\frac{1}{x(\sqrt{x^2+1}-x)} &= \lim_{n\to+\infty}\frac{1}{x(\sqrt{x^2+1}-x)}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\
&= \lim_{n\to+\infty}\frac{\sqrt{x^2+1}+x}{x(x^2+1-x^2)} = \lim_{n\to+\infty}\frac{\sqrt{x^2+1}+x}{x}\\
&= \lim_{n\to+\infty}\Big(\sqrt{1+\frac{1}{x^2}}+1\Big) = 2
\end{aligned}$$

4. *a)* Funkcija je neprekidna u točki ako je njezin limes u toj točki jednak njezinoj vrijednosti u toj točki, tj. ako je

$$\lim_{x\to a} f(x) = f(a).$$

*b)* Funkcija $f$ je neprekinuta u svakoj točki $a \neq 0$. U točki nula imamo:

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(x^2+\cos x) = 1 = f(0).$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\frac{\ln(1+ax)}{x} = a\lim_{x\to 0-}\frac{\ln(1+ax)}{ax} = a.$$

$f$ neprekinuta u nuli $\Rightarrow \lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0) \Rightarrow a = 1.$

## Grupa B:

1. *a)*

$$\lim_{n\to\infty}\left(\frac{2n+2}{n+2}\right)^{\frac{n+2}{n}} = \left(\lim_{n\to\infty}\frac{2n+2}{n+2}\right)^{\lim_{n\to\infty}\frac{n+2}{n}} = \left(\lim_{n\to\infty}\frac{2+\frac{2}{n}}{1+\frac{2}{n}}\right)^{\lim_{n\to\infty}(1+\frac{2}{n})} = 2^1 = 2$$

*b)*

$$\lim_{n\to\infty}\left(\frac{2n+2}{n+2}\right)^{n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{2n+1}\right)^{2n+1}\right]^{\frac{n}{2n+1}} = e^{\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}} = e^{\frac{1}{2}} = \sqrt{e}.$$

2. *i)* Niz $(a_n)$ je rastući niz:
Baza: $a_2 = \frac{13}{4} > 1 = a_1$
Pretpostavka indukcije: $a_n > a_{n-1}$
Korak: $a_n > a_{n-1} \Rightarrow \frac{1}{4}a_n > \frac{1}{4}a_{n-1} \Rightarrow \frac{1}{4}a_n + 3 > \frac{1}{4}a_{n-1} + 3 \Rightarrow a_{n+1} > a_n$

*ii)* Niz $(a_n)$ je ograničen odozgo:
Baza: $a_1 < 4$
Pretpostavka indukcije: $a_n < 4$
Korak: $a_{n+1} = \frac{1}{4}a_n + 3 < \frac{1}{4}\cdot 4 + 3 = 4$

*iii)* Budući da je monoton i ograničen, niz $(a_n)$ je konvergentan i postoji $L = \lim_{n\to\infty} a_n$. Vrijedi:

$$a_{n+1} = \frac{1}{4}a_n+3 \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{4}a_n+3\right) \Rightarrow L = \frac{1}{4}L+3 \Rightarrow L = 4.$$

3. *a)*

$$\lim_{n\to+\infty}\frac{1}{\sqrt{x^2+x+1}-x} = \lim_{n\to+\infty}\frac{1}{\sqrt{x^2+x+1}-x}\cdot\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} = \lim_{n\to+\infty}\frac{\sqrt{x^2+x+1}+x}{x^2+x+1-x^2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{n\to+\infty}\frac{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}{1+\frac{1}{x}} = \frac{\sqrt{1}+1}{1} = 2.$$

*b*)

$$\begin{aligned}\lim_{x\to+\infty}\frac{x+1}{\sqrt{x^2+1}+x} &= \lim_{x\to+\infty}\frac{x+1}{\sqrt{x^2+1}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\\ &= \lim_{x\to+\infty}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}=\frac{1}{2}.\end{aligned}$$

4. *a*) Funkcija je neprekidna u točki ako je njezin limes u toj točki jednak njezinoj vrijednosti u toj točki, tj. ako je

$$\lim_{x\to a} f(x) = f(a).$$

*b*) Funkcija $f$ je neprekinuta u svakoj točki $a \neq 0$. U točki nula imamo:

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x - e^x) = -1 = f(0).$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\arcsin{(ax)}}{x} = a \lim_{x\to 0-} \frac{\arcsin{(ax)}}{ax} = a.$$

$f$ neprekinuta u nuli $\Rightarrow \lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0) \Rightarrow a = -1.$