Rješenja ponovljenog prvog međuispita iz Matematike 1

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1. $|z| = \dfrac{1}{|z|} \Rightarrow |z| = 1$ što zajedno s $|z-1| = 1$ daje da su rješenja dvije točke u kompleksnoj ravnini koje se nalaze na presjeku kružnica radijusa 1 oko ishodišta i oko točke 1.

   Dakle, $Re(z_{1,2}) = \dfrac{1}{2}$, $Im(z_{1,2}) = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2} = \pm\dfrac{\sqrt{3}}{2}$.

   $z_1 = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$, $z_2 = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

2. (a) Predavanja.

   (b) $a = \dfrac{\pi}{5}$, $Im(f) = [0, 4]$.

3. Domena: $\left|\dfrac{\pi}{x}\right| \leq 1 \Rightarrow |x| \geq \pi \Rightarrow D(f) = \langle -\infty, -\pi] \cup [\pi, \infty\rangle$.

   $\arcsin\dfrac{\pi}{x} = \dfrac{\pi}{6} \Rightarrow \dfrac{\pi}{x} = \dfrac{1}{2} \Rightarrow x = 2\pi$.

4. $\det \mathbf{A} = (a+3)(a-1)^3 \Rightarrow a \in \mathbb{R}\backslash\{-3, 1\}$.

5. (a) $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$, jer je
   $\mathbf{ABCC}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{I}$.

   (b) $(\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{C})^{-1} = \mathbf{C}^{-1}\mathbf{BA}$

   $$\mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{C}^{-1}\mathbf{BA} = \begin{bmatrix} 4 & -3 & 1 \\ -5 & 4 & -2 \\ 2 & -1 & 1 \end{bmatrix}$$

6. $(\mathbf{AB} + \mathbf{BA})^\top = (\mathbf{AB})^\top + (\mathbf{BA})^\top = \mathbf{B}^\top\mathbf{A}^\top + \mathbf{A}^\top\mathbf{B}^\top = \mathbf{BA} + \mathbf{AB} = \mathbf{AB} + \mathbf{BA}$

7. Matrica sustava je $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & \lambda \end{bmatrix}$ što je ekvivalentno s $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & \lambda+2 \end{bmatrix}$.

   Sada vidimo da za $\lambda = -2$ imamo $x = -3z$ i $y = 2z$ za $z \in \mathbb{R}$, tj. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, $\alpha \in \mathbb{R}$.

   Za $\lambda \neq -2$ slijedi $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

8. (a) Predavanja.

(b) $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 - \lambda + 1 = -(\lambda - 1)(\lambda^2 + 1).$

Jedina realna svojstvena vrijednost je $\lambda = 1$, a pripadni svojstveni vektori su $\alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ za $\alpha \in \mathbb{R} \backslash \{0\}$.