Rješenja 2. školske zadaće za grupe 1 i 5
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Podgrupa A

1. a) Gomilište niza $(a_n)_{n\in\mathbb{N}}$ je broj A takav da se za svaki $\varepsilon > 0$ u ε - okolini broja A nalazi beskonačno mnogo članova niza $(a_n)_{n\in\mathbb{N}}$.

b) Gomilišta su limesi dva podniza, na parnim i neparnim pozicijama: $\frac{3}{2}, -\frac{3}{2}$.

2. $a_2 = \frac{3}{2},\ a_3 = \frac{5}{4},\ a_4 = \frac{9}{8}, \ldots$. Indukcijom dokazujemo da je $a_n = \frac{2^{n-1}+1}{2^{n-1}} = 1 + \frac{1}{2^{n-1}}$. Baza: $n = 1$, $a_1 = 1 + \frac{1}{2^0} = 2$. Pretpostavimo da je $a_n = 1 + \frac{1}{2^{n-1}}$ za neki $n \in \mathbb{N}$. Za $n + 1$ imamo $a_{n+1} = \frac{1}{2}(a_n + 1) = \frac{1}{2}(1 + \frac{1}{2^{n-1}} + 1) = 1 + \frac{1}{2\cdot 2^{n-1}} = 1 + \frac{1}{2^n}$. Sada je $\lim_{n\to\infty} a_n = 1 + \lim_{n\to\infty} \frac{1}{2^{n-1}} = 1$.

3. $$\lim_{x\to+\infty}(\frac{x}{x+5})^x = \lim_{x\to\infty} \frac{1}{(\frac{x+5}{x})^x} = \frac{1}{\lim_{x\to\infty}(1+\frac{5}{x})^x} = \frac{1}{(\lim_{x\to\infty}(1+\frac{1}{\frac{x}{5}})^{\frac{x}{5}})^5} = \frac{1}{e^5}.$$

4. a) $$\lim_{x\to 0}(\frac{e^{3x}-e^{2x}}{x}) = \lim_{x\to 0}(\frac{e^{3x}-1-e^{2x}+1}{x}) = \lim_{x\to 0}(\frac{e^{3x}-1}{x\cdot\frac{3}{3}}) - \lim_{x\to 0}(\frac{e^{2x}-1}{x\cdot\frac{2}{2}}) =$$
$$3\lim_{x\to 0}(\frac{e^{3x}-1}{3x}) - 2\lim_{x\to 0}(\frac{e^{2x}-1}{2x}) = 3\cdot 1 - 2\cdot 1 = 1.$$

b) $$\lim_{x\to+\infty} \frac{x^2}{2x+1}\mathrm{arctg}(\frac{1}{x}) = \lim_{x\to+\infty} \frac{x^2}{2x+1}\mathrm{arctg}(\frac{1}{x}) \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x}{2x+1} \cdot \frac{\mathrm{arctg}(\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x}{2x+1} \cdot \lim_{x\to+\infty} \frac{\mathrm{arctg}(\frac{1}{x})}{\frac{1}{x}} = [y = \frac{1}{x}] = \frac{1}{2}\cdot\lim_{y\to 0} \frac{\mathrm{arctg}(y)}{y} = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

Podgrupa B

1. a) Limes niza $(a_n)_{n\in\mathbb{N}}$ je broj L (ako takav postoji) sa svojstvom da za svaki $\varepsilon > 0$ postoji $n_0 \in \mathbb{N}$ takav da $n \geq n_0 \;\Rightarrow |a_n - L| < \varepsilon$.

b) $\lim_{n\to\infty} a_n = 0$.

2. $a_2 = \frac{4}{3},\; a_3 = \frac{10}{9},\; a_4 = \frac{28}{27}, \ldots$. Indukcijom dokazujemo da je $a_n = \frac{3^{n-1}+1}{3^{n-1}} = 1 + \frac{1}{3^{n-1}}$. Baza: $n = 1$, $a_1 = 1 + \frac{1}{3^0} = 2$. Pretpostavimo da je $a_n = 1 + \frac{1}{3^{n-1}}$ za neki $n \in \mathbb{N}$. Za $n+1$ imamo $a_{n+1} = \frac{1}{3}(a_n + 2) = \frac{1}{3}(1 + \frac{1}{3^{n-1}} + 2) = 1 + \frac{1}{3\cdot 3^{n-1}} = 1 + \frac{1}{3^n}$. Sada je $\lim_{n\to\infty} a_n = 1 + \lim_{n\to\infty} \frac{1}{3^{n-1}} = 1$.

3. $\lim\limits_{x\to+\infty}(\sqrt{x^2+x}-x) = \lim\limits_{x\to+\infty}(\sqrt{x^2+x}-x)\cdot\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim\limits_{x\to+\infty}\dfrac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim\limits_{x\to+\infty}\dfrac{x}{\sqrt{x^2+x}+x} : \dfrac{x}{x} = \lim\limits_{x\to+\infty}\dfrac{1}{\sqrt{1+\frac{1}{x}}+1} = \dfrac{1}{2}.$

4. a) $\lim\limits_{x\to+\infty}(x[\ln(x+1)-\ln x]) = \lim\limits_{x\to+\infty}\dfrac{\ln(\frac{x+1}{x})}{\frac{1}{x}} = \{y = \dfrac{1}{x}\} = \lim\limits_{y\to 0}\dfrac{\ln(1+y)}{y} = 1.$

b) $\lim\limits_{x\to+\infty}\dfrac{x^3}{3x^2+1}\sin(\dfrac{1}{x}) = \lim\limits_{x\to+\infty}\dfrac{x^3}{3x^2+1}\sin(\dfrac{1}{x})\cdot\dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to+\infty}\dfrac{x^2}{3x^2+1}\cdot\dfrac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim\limits_{x\to+\infty}\dfrac{x^2}{3x^2+1}\cdot\lim\limits_{x\to+\infty}\dfrac{\sin(\frac{1}{x})}{\frac{1}{x}} = [y = \dfrac{1}{x}] = \dfrac{1}{3}\cdot\lim\limits_{y\to 0}\dfrac{\sin y}{y} = \dfrac{1}{3}\cdot 1 = \dfrac{1}{3}.$