Rješenja 2. međuispita iz Matematike 1
17.5.2006.

1. [2 boda] **Monotonost:** niz je rastući; dokaz mat. indukcijom po $n \in \mathbb{N}$;
Ograničenost: tvrdimo da je niz ograničen $0 < a_n < 2$; dokaz mat. indukcijom po $n \in \mathbb{N}$;
Niz je monoton i ograničen $\Rightarrow$ niz je konvergentan i $\exists L = \lim\limits_{n\to\infty} a_n$;
$a_{n+1} = \sqrt{2a_n} \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{2a_n} \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \sqrt{2 \lim\limits_{n\to\infty} a_n} \Leftrightarrow$
$\Leftrightarrow L = \sqrt{2L} \Leftrightarrow L = 0$ ili $L = 2$. Slučaj $L = 0$ otpada, jer je $a_n > 0$.

2. [2 boda] $\lim\limits_{x\to\infty} \left(\dfrac{2x+5}{2x+8}\right)^x = e^{\lim\limits_{x\to\infty} \frac{-3x}{2x+8}} = e^{-\frac{3}{2}}$

3. [2 boda] a) $\lim\limits_{x\to+\infty} (\sqrt{1+x+x^2} - \sqrt{1-x+x^2}) \cdot \dfrac{(\sqrt{1+x+x^2} + \sqrt{1-x+x^2})}{(\sqrt{1+x+x^2} + \sqrt{1-x+x^2})} =$
$= \lim\limits_{x\to+\infty} \dfrac{2x}{2x} = 1$
b) $\lim\limits_{x\to-\infty} (\sqrt{1+x+x^2} - \sqrt{1-x+x^2}) =$ supst: $x = -t$ $= \lim\limits_{t\to+\infty} (\sqrt{1-t+t^2} - \sqrt{1+t+t^2}) \cdot \dfrac{(\sqrt{1-t+t^2} + \sqrt{1+t+t^2})}{(\sqrt{1-t+t^2} + \sqrt{1+t+t^2})} = \lim\limits_{t\to+\infty} (-\dfrac{2t}{2t}) = -1$

4. [2 boda] a)Neka je $x_0 \in \mathbb{R}$. Funkcija $y = f(x)$ je neprekinuta u točki $x_0 \in D_f$ ako je $\lim\limits_{x\to x_0} f(x) = f(x_0)$.
ili
Neka je $x_0 \in \mathbb{R}$. Funkcija $y = f(x)$ je neprekinuta u točki x_0 ako je $x_0 \in D_f$ i ako $\forall \varepsilon > 0\ \exists \delta > 0$ takav da vrijedi:

$$\forall x \in D_f,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

b)Uvjet: $\lim\limits_{x\to0+} f(x) = f(0) = \lim\limits_{x\to0-} f(x);\ f(0) = a;\ \lim\limits_{x\to0+} f(x) = \lim\limits_{x\to0+} \dfrac{x}{\sin(3x)} =$
$= \dfrac{1}{3} \Rightarrow a = \dfrac{1}{3}.$

5. [2 boda] $(\sin x)' = \lim\limits_{h\to0} \dfrac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to0} \dfrac{2}{h} \cdot \sin\dfrac{h}{2} \cdot \cos(x + \dfrac{h}{2}) =$
$= \lim\limits_{h\to0} \left[\dfrac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \cos(x + \dfrac{h}{2})\right] = \lim\limits_{h\to0} \dfrac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \lim\limits_{h\to0} \cos(x + \dfrac{h}{2}) = \cos\left(\lim\limits_{h\to0} (x + \dfrac{h}{2})\right) = \cos x$

6. [3 boda]

$$(f^{-1}(y))' = \frac{1}{f'(x)}$$

$y = \sin x,\ x = \text{arcsin}y$
$(\text{arcsin}y)' = \dfrac{1}{(\sin x)'} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1 - \sin^2 x}} = \dfrac{1}{\sqrt{1 - y^2}}$

7. [2 boda] $(4, \dfrac{\pi}{3});\ y' = \dfrac{1}{2\sqrt{x}} \cdot \text{arcsin}(\dfrac{1}{\sqrt{x}}) + \sqrt{x} \dfrac{1}{\sqrt{1 - \frac{1}{x}}} \cdot -\dfrac{1}{2x\sqrt{x}};$
$t.....y - \dfrac{\pi}{3} = (\dfrac{\pi}{24} - \dfrac{\sqrt{3}}{12})(x - 4)$

8. [2 boda] $y'(1) = -\frac{5}{8};\ y''(1) = \frac{111}{256}$

9. [2 boda] a) Neka je $f : [a, b] \to \mathbb{R}$ neprekinuta funkcija diferencijabilna na (a, b). Onda postoji $c \in (a, b)$ takav da je

$$f(b) - f(a) = f'(c)(b - a).$$

b) $\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2} \Leftrightarrow c_{1,2} = \pm\sqrt{1 - (\frac{2}{\pi})^2}$

10. [2 boda] $\lim_{x\to\infty} x(\ln(e + \frac{1}{e} - 1)) = (\infty \cdot 0) = \lim_{x\to\infty} \frac{\ln(e + \frac{1}{x} - 1)}{\frac{1}{x}} = (\frac{0}{0}) =$

$= L'H = \lim_{x\to\infty} \frac{1}{e + \frac{1}{x}} = \frac{1}{e}$